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## MEAN VALUES WITH CUBIC CHARACTERS

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ABSTRACT. We investigate various mean value problems involving order three primitive Dirichlet characters. In particular, we obtain an asymptotic formula for the first moment of central values of the Dirichlet  $L$ -functions associated to this family, with a power saving in the error term. We also obtain a large-sieve type result for order three (and six) Dirichlet characters.

## 1. INTRODUCTION AND MAIN RESULTS

Dirichlet characters of a given order appear naturally in many applications in number theory. The quadratic characters have seen a lot of attention due to attractive questions to ranks of elliptic curves, class numbers, etc. In particular, we are interested in mean values of  $L$ -functions twisted by characters of fixed order, and also large sieve-type inequalities for these characters.

In this work we consider various analytic problems involving twists by order three (and order six, to a lesser extent) Dirichlet characters. Our first result on such  $L$ -functions is the following

**Theorem 1.1.** *Let  $w : (0, \infty) \rightarrow \mathbb{R}$  be a smooth, compactly supported function. Then*

$$(1) \quad \sum_{(q,3)=1} \sum_{\substack{\chi \pmod{q} \\ \chi^3 = \chi_0}}^* L\left(\frac{1}{2}, \chi\right) w\left(\frac{q}{Q}\right) = cQ\widehat{w}(0) + O(Q^{13/14+\varepsilon}),$$

where  $c > 0$  is a constant that can be given explicitly in terms of an Euler product, and  $\widehat{w}$  is the Fourier transform of  $w$ . Here the  $*$  on the sum over  $\chi$  restricts the sum to primitive characters.

This result is most similar (in terms of method of proof) to the main result of [L], who considered the analogous mean value but for the case of cubic Hecke  $L$ -functions on  $\mathbb{Q}(\omega)$ ,  $\omega = e^{2\pi i/3}$ . Our problem has new analytic difficulties that can be traced to the fact that Poisson summation has quality related to the degree of  $[\mathbb{Q}(\omega) : \mathbb{Q}] = 2$ . In effect, it is more difficult to study twists of rational integers by cubic characters than to study cubic twists of elements of  $\mathbb{Z}[\omega]$ , a fact which appears somewhat paradoxical. In some sense, the degree of the extension causes this problem to be analytically somewhat similar to a second moment (in terms of the strength of Poisson summation). As an aside, it should not be surprising that  $c > 0$  here since the set of central values is invariant under complex conjugation.

Other authors ([FaHL], [FrHL], [Di], [BFH], ...) have considered cubic and higher order twists using multiple Dirichlet series. However, the method using the metaplectic Eisenstein series currently requires the ground field to contain the  $l$ -th roots of unity (supposing one is twisting by order  $l$  Hecke characters). It is interesting that this is an obstruction to proving Theorem 1.1 that is seemingly completely different than the analytic issues with Poisson summation. Diaconu and Tian [DT] have developed analytic properties of a multiple Dirichlet series that potentially has applications to the first moment considered in our Theorem 1.1. By taking  $r = 3$ ,  $F = \mathbb{Q}$ , and  $L = \mathbb{Q}(\omega)$  (in their Section 3) and interchanging the

orders of summation they obtain a multiple Dirichlet series which essentially parameterizes the unsieved first moment above. We thank an anonymous referee for pointing this out to us. However, it is not clear if our Theorem 1.1 can be obtained from [DT].

We furthermore show

**Theorem 1.2.** *Let  $Q \geq 1$ . Then we have*

$$(2) \quad \sum_{q \leq Q} \sum_{\substack{\chi \pmod{q} \\ \chi^3 = \chi_0}}^* |L(1/2, \chi)|^2 \ll Q^{6/5+\varepsilon}.$$

Consequently, we have

**Corollary 1.3.** *There exist infinitely many primitive Dirichlet characters  $\chi$  of order 3 such that  $L(1/2, \chi) \neq 0$ . More precisely, the number of such characters with conductor  $\leq Q$ , say  $N_3(Q)$ , is  $\gg Q^{4/5-\varepsilon}$ .*

Studies of the moments and nonvanishing of cubic twists of elliptic curves using random matrix theory have been carried out in [DFK]. Nonvanishing of cubic twists of elliptic curves using algebraic methods has been undertaken by [FKK].

It is this upper bound on the second moment, combined with Theorem 1.1 and a standard application of Cauchy-Schwarz that gives Corollary 1.3:

$$(3) \quad Q \ll \sum_{q \leq Q} \sum_{\substack{\chi \pmod{q} \\ \chi^3 = \chi_0}}^* |L(\tfrac{1}{2}, \chi)| \leq N_3(Q)^{1/2} \left( \sum_{q \leq Q} \sum_{\substack{\chi \pmod{q} \\ \chi^3 = \chi_0}}^* |L(\tfrac{1}{2}, \chi)|^2 \right)^{1/2} \leq N_3(Q)^{1/2} Q^{3/5+\varepsilon}.$$

We shall establish Theorem 1.2 by using the following large sieve-type result with cubic Dirichlet characters.

**Theorem 1.4.** *Let  $(a_m)_{m \in \mathbb{N}}$  be an arbitrary sequence of complex numbers. Then*

$$(4) \quad \sum_{Q < q \leq 2Q} \sum_{\substack{\chi \pmod{q} \\ \chi^3 = \chi_0}}^* \left| \sum_{\substack{M < m \leq 2M \\ (m, 3)=1}}^* a_m \chi(m) \right|^2 \\ \ll (QM)^\varepsilon \min \{ Q^{5/3} + M, Q^{4/3} + Q^{1/2}M, Q^{11/9} + Q^{2/3}M, Q + Q^{1/3}M^{5/3} + M^{12/5} \} \times \\ \sum_{\substack{M < m \leq 2M \\ (m, 3)=1}}^* |a_m|^2,$$

where the star attached to the sum over  $m$  indicates that  $m$  runs over square-free integers.

For comparison, we note that the ordinary large sieve gives the following weaker upper bounds

$$(5) \quad \sum_{q \leq Q} \sum_{\substack{\chi \pmod{q} \\ \chi^3 = \chi_0}}^* |L(\tfrac{1}{2}, \chi)|^8 \ll Q^{2+\varepsilon}, \quad \sum_{q \leq Q} \sum_{\substack{\chi \pmod{q} \\ \chi^3 = \chi_0}}^* |L(\tfrac{1}{2}, \chi)|^2 \ll Q^{\frac{5}{4}+\varepsilon},$$

$$(6) \quad \sum_{q \leq Q} \sum_{\substack{\chi \pmod{q} \\ \chi^3 = \chi_0}}^* |L(\tfrac{1}{2}, \chi)| \ll Q^{\frac{9}{8}+\varepsilon}.$$

The 8th moment is deduced simply by embedding the family of cubic characters into the family of all Dirichlet characters of conductor  $\leq Q$ , and using the known bound on the 8th

moment of the  $L$ -functions associated to the latter family. The estimates for the first and second moments follow by Cauchy's inequality.

We point to [E], Section 7, for some early large sieve-type results on general  $r$ -th order characters. Related results to Theorem 1.4 are Heath-Brown's quadratic large sieve [Hea1], which states

$$(7) \quad \sum_{q \leq Q} \sum_{\substack{\chi \pmod{q} \\ \chi^2 = \chi_0}}^* \left| \sum_{m \leq M} \mu^2(m) a_m \chi(m) \right|^2 \ll (Q + M)(QM)^\varepsilon \sum_{m \leq M}^* \mu^2(m) |a_m|^2,$$

and his cubic large sieve [Hea2]

$$(8) \quad \sum_{\substack{n \in \mathbb{Z}[\omega] \\ N(n) \leq N}}^* \left| \sum_{\substack{m \in \mathbb{Z}[\omega] \\ N(m) \leq M}}^* a_m \left( \frac{m}{n} \right)_3 \right|^2 \ll (M + N + (MN)^{2/3})(MN)^\varepsilon \sum_{m \leq M} |a_m|^2,$$

where the stars indicate that  $m, n$  run over square-free elements of  $\mathbb{Z}[\omega]$  that are congruent to 1 (mod 3). Both of these results are proved with a recursive use of Poisson summation; the latter result is not optimal due to the term  $(MN)^{2/3}$ . The presence of this term can be traced to a loss in the treatment of the square-free condition, which becomes more difficult with cubic characters. Our method of proof of Theorem 1.4 uses (8) (after some transformations), and avoids a direct recursion. One of the new difficulties with treating cubic Dirichlet characters is the asymmetry between  $\chi$  and  $m$  (essentially  $\chi$  is parametrized by elements of  $\mathbb{Z}[e^{2\pi i/3}]$ ).

It is of great interest to extend these results to higher-order characters and to different number fields. In general we wish to understand these families of twists in the Katz-Sarnak sense [KS1] [KS2]. The most essential analytic issue is what is the degree of the field extension  $\mathbb{Q}(e^{2\pi i/l})/\mathbb{Q}$ . It is plausible that our methods could generalize to  $l = 4, 6$  since for these cases this degree is also 2. However, for the application of the central values of  $L$ -functions, we also require estimates for the sum of  $l$ -th order Gauss sums, which become somewhat worse as  $l$  increases (e.g. see Proposition 1 of [P2]).

We can generalize some of our results to sextic characters.

**Theorem 1.5.** *Theorems 1.2 and 1.4 remain valid when the condition  $\chi^3 = \chi_0$  is replaced by the weaker condition  $\chi^6 = \chi_0$ .*

A motivating example for this generalization to sextic twists is to understand the behavior of the family of elliptic curves

$$(9) \quad y^2 = x^3 + b,$$

where  $b \in \mathbb{Z}$ . These curves have complex multiplication by  $\mathbb{Q}(\omega)$ , and have  $L$ -functions that can be expressed using the sextic residue symbol  $(4b/n)_6$ , where  $n$  runs over elements of  $\mathbb{Z}[\omega]$ . The study of this family of  $L$ -functions clearly leads to double sums of the form addressed in Theorem 1.5.

Another potential application of Theorem 1.5 is in the study of the low-lying zeros of the family of all elliptic curves (which has applications to the average rank of the family). In [Y] the second named author related the problem to a certain complicated character sum which contains a sum over Dirichlet characters with moduli in certain intervals. In particular, cubes of characters appear, and the analysis of the problem naturally treats the cubic characters differently than for non-cubic characters. The cubic character contribution

has been the limit of the unconditional method to obtaining the asymptotic of the 1-level density of this family [BL]. Preliminary calculations of the first named author and L. Zhao indicate that Theorem 1.5 is potentially useful for improving the estimations of the cubic character contribution.

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## 2. PRELIMINARIES

We begin with proving Theorem 1.1. In the following, we provide the required tools.

**2.1. The approximate functional equation.** Using an approximate functional equation (see Theorem 5.3 of [IK]), we have

**Proposition 2.1.** *Let  $\chi$  be an odd primitive Dirichlet character  $\chi$  of conductor  $q$ , and make the following definitions: Let*

$$(10) \quad V_\alpha(x) = \frac{1}{2\pi i} \int_{(1)} \frac{G(s)}{s} g_\alpha(s) x^{-s} ds,$$

where

$$(11) \quad g_\alpha(s) = \pi^{-s/2} \frac{\Gamma\left(\frac{\frac{3}{2} + \alpha + s}{2}\right)}{\Gamma\left(\frac{\frac{3}{2} + \alpha}{2}\right)}.$$

Furthermore, set

$$(12) \quad X_\alpha = \left(\frac{q}{\pi}\right)^{-\alpha} \frac{\Gamma\left(\frac{\frac{3}{2} - \alpha}{2}\right)}{\Gamma\left(\frac{\frac{3}{2} + \alpha}{2}\right)},$$

where  $\epsilon(\chi) = i^{-1} q^{-1/2} \tau(\chi)$  is essentially the (normalized) Gauss sum associated to  $\chi$ . Finally let  $A$  and  $B$  be positive real numbers such that  $AB = q$ . Then

$$(13) \quad L\left(\frac{1}{2} + \alpha, \chi\right) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^{\frac{1}{2} + \alpha}} V_\alpha\left(\frac{m}{A}\right) + \epsilon(\chi) X_\alpha \sum_{m=1}^{\infty} \frac{\bar{\chi}(m)}{m^{\frac{1}{2} - \alpha}} V_{-\alpha}\left(\frac{m}{B}\right).$$

**2.2. Classification of the cubic characters.** In the following, we shall classify all the primitive cubic characters of conductor  $q$  coprime to 3. The answer is that every such character is of the form  $m \rightarrow \left(\frac{m}{n}\right)_3$  for some  $n \in \mathbb{Z}[\omega]$ ,  $n \equiv 1 \pmod{3}$ ,  $n$  not divisible by any rational primes, with norm  $N(n) = q$ , where  $\omega = e^{2\pi i/3}$ . This analysis can also be found in [DFK] e.g. but we shall present this here for completeness.

To classify the characters we use an approach similar to that of [Da], Chapter 5. By multiplicativity, it suffices to consider the case that  $q = p^a$  is a prime power. It is not hard to show that there is a primitive character of conductor  $p$  if and only if  $p \equiv 1 \pmod{3}$ , in which case there are exactly two such characters  $\pmod{p}$ , each being the square of the

other. If  $a \geq 2$  and  $p \neq 3$  then there is no primitive character of order 3, as any character of order 3 must be induced from one  $(\bmod p)$ .

These cubic characters are intimately connected with the cubic residue symbol in the ring  $\mathbb{Z}[\omega]$ . If  $p \equiv 1 \pmod{3}$  then  $p = \pi\bar{\pi}$  with  $N(\pi) = p$ , and then there is an associated cubic character  $n \rightarrow (\frac{n}{\pi})_3$ . This cubic character is defined by the conditions  $(\frac{n}{\pi})_3 \equiv n^{\frac{N(\pi)-1}{3}} \pmod{\pi}$ , with  $(\frac{n}{\pi})_3 \in \{1, \omega, \omega^2\}$ . It follows directly from the definition that for  $n \in \mathbb{Z}$ ,  $(\frac{n}{\pi})_3 = \overline{(\frac{n}{\pi})_3}$ . Thus for prime conductor  $p$  there is a one-to-one correspondence between primitive Dirichlet characters of order 3 and conductor  $p$ , and cubic residue symbols  $\chi_\pi$  with  $N(\pi) = p$ . By multiplicativity we extend this one-to-one correspondence to squarefree  $q$  and elements  $n$  of  $\mathbb{Z}[\omega]$  such that  $N(n) = q$ . It is easy to see that  $N(n)$  is squarefree (as an element of  $\mathbb{Z}$ ) if and only if  $n$  is squarefree (as an element of  $\mathbb{Z}[\omega]$ ) and  $n$  has no rational prime divisor.

Thus we have that every primitive cubic Dirichlet character of conductor coprime to 3 is of the form  $\chi_n : m \rightarrow (\frac{m}{n})_3$ , where  $n \in \mathbb{Z}[\omega]$  is squarefree, has no rational prime divisor, and  $n \equiv 1 \pmod{3}$ .

Recall that the cubic reciprocity law states that

$$(14) \quad \left(\frac{m}{n}\right)_3 = \left(\frac{n}{m}\right)_3$$

provided that  $m, n \equiv \pm 1 \pmod{3}$ . The supplement states that if  $\pi = 1 + 3a + 3b\omega$ , where  $a, b \in \mathbb{Z}$ , then

$$(15) \quad \left(\frac{1-\omega}{\pi}\right)_3 = \omega^a.$$

Note  $3 = -\omega^2(1-\omega)^2$ , and that  $N(\pi) = (1+3a)^2 - (1+3a)3b + 9b^2 \equiv 1 + 3a + 3b \pmod{9}$ . Therefore, a simple calculation shows

$$(16) \quad \left(\frac{\omega}{\pi}\right)_3 = \omega^{2a+2b},$$

and hence

$$(17) \quad \left(\frac{3}{\pi}\right)_3 = \omega^b.$$

It follows easily that for any  $n \equiv 1 \pmod{3}$ ,  $n \in \mathbb{Z}[\omega]$ , written in the form  $n = 1 + 3c + 3d\omega$ , then  $(3/n)_3 = \omega^d$ . In particular, for  $n \equiv 1 \pmod{3}$ , we have  $(3/n)_3 = 1$  if and only if  $n \equiv 1, 4, 7 \pmod{9}$ , and in general  $(3/n)_3$  only depends on  $n \pmod{9}$ .

**2.3. On the Gauss sums.** It turns out that the Gauss sum associated to the Dirichlet character  $\chi_n$  (on  $\mathbb{Z}$ ) is the same one as the corresponding Hecke character (on  $\mathbb{Z}[\omega]$ ). We now prove this important fact. Recall that  $n$  is  $\equiv 1 \pmod{3}$  and has no rational prime divisor, so  $(n, \bar{n}) = 1$ . By definition,

$$(18) \quad \tau(\chi_n) = \sum_{1 \leq x \leq N(n)} \left(\frac{x}{n}\right)_3 e^{\frac{2\pi i x}{N(n)}}.$$

Now write  $x = y\bar{n} + \bar{y}n$ , where  $y$  varies over a set of representatives in  $\mathbb{Z}[\omega] \pmod{n}$ , and here  $\bar{n}$  is the complex conjugate of  $n$ . It is easy to see that as  $y$  varies  $(\bmod n)$ ,  $x$  varies  $(\bmod N(n))$ , using that  $\bar{x} = x$  and the Chinese Remainder Theorem. We then see

$$(19) \quad \tau(\chi_n) = \sum_{y \pmod{n}} \left(\frac{y\bar{n}}{n}\right)_3 e^{2\pi i \left(\frac{y}{n} + \frac{\bar{y}}{n}\right)}.$$

It is a consequence of cubic reciprocity that  $(\frac{\bar{n}}{n})_3 = 1$  for  $n \equiv 1 \pmod{3}$ , so we have

$$(20) \quad \tau(\chi_n) = g(n),$$

where  $g(n)$  is the normalized Gauss sum as defined in [H-BP], for instance (note that they use the notation  $e(z) = e^{2\pi i(z+\bar{z})}$ , whereas we shall use  $e(z) = e^{2\pi iz}$ , and let  $\tilde{e}(z)$  correspond to their  $e(z)$ ). One has

$$(21) \quad g(n)^3 = \mu(n)N(n)n,$$

whence one derives the pleasant fact that  $g(n)$  vanishes unless  $n$  is squarefree. More generally, we set

$$(22) \quad g(r, n) = \sum_{x \pmod{n}} \left(\frac{x}{n}\right)_3 \tilde{e}\left(\frac{rx}{n}\right).$$

We also collect the following elementary relations, for which see [H-BP], pp.123-124.

$$(23) \quad g(rs, n) = \overline{\left(\frac{s}{n}\right)_3} g(r, n), \quad \text{if } (s, n) = 1.$$

Furthermore, if  $(n_1, n_2) = 1$  then

$$(24) \quad g(r, n_1 n_2) = \overline{\left(\frac{n_1}{n_2}\right)_3} g(r, n_1) g(r, n_2)$$

$$(25) \quad = g(n_2 r, n_1) g(r, n_2).$$

We also compute, for  $\pi$  prime in  $\mathbb{Z}[\omega]$ ,  $k \geq 1$ ,

$$(26) \quad g(\pi^2, \pi^k) = \begin{cases} -N(\pi^2), & k = 3, \\ 0, & \text{otherwise.} \end{cases}$$

In addition, we shall require the fact that

$$(27) \quad g(r, n) = 0, \quad \text{if } \pi^2 | n, \pi \nmid r,$$

which follows immediately from (21) and (23).

### 3. OUTLINE AND PRELIMINARIES FOR THE PROOF OF THEOREM 1.1

We first note that by the discussion in Section 2.2,

$$(28) \quad \sum_{\substack{(q,3)=1 \\ \chi^3=\chi_0}} \sum_{\chi \pmod{q}}^* L\left(\frac{1}{2}, \chi\right) w\left(\frac{q}{Q}\right) = \sum'_{n \equiv 1 \pmod{3}} L\left(\frac{1}{2}, \chi_n\right) w\left(\frac{N(n)}{Q}\right)$$

where the sums on the right-hand side run over squarefree elements  $n$  of  $\mathbb{Z}[\omega]$  that have no rational prime divisor.

Our goal is to obtain a kind of recursive estimate for the central values. To do so, we need to generalize our setup somewhat, and consider sums of the form

$$(29) \quad M_\alpha(w, X, r, \xi, \beta, a) = \sum'_{\substack{(n, r\bar{r}\xi)=1 \\ n \equiv \beta \pmod{9}}} L\left(\frac{1}{2} + \alpha, \chi_{nr}\right) \chi_{nr}(a) w\left(\frac{N(n)}{X}\right),$$

where  $\chi_r$  is primitive,  $a \in \mathbb{N}$ ,  $\beta \in \mathbb{Z}[\omega]/(9)$ ,  $\beta \equiv 1 \pmod{3}$ , and  $\alpha$  is a complex number in the box  $0 \leq \operatorname{Re}(\alpha) \leq \varepsilon$ ,  $\operatorname{Im}(\alpha) \ll X^\varepsilon$ , where  $\varepsilon$  is a sufficiently small positive constant. Furthermore, the prime on the sum over  $n$  indicates that  $n$  is squarefree and has no rational prime divisor.

The conditions on the summation over  $n$  ensure that  $\chi_{nr}$  is a primitive order 3 Dirichlet character of conductor  $N(nr)$ . Our moment of interest corresponds to the case  $\xi = r = a = 1$ , but the recursive analysis of this moment leads to these more general expressions. We also assume that  $\log N(ar\xi) \ll \log 2X$ , so  $N(ar\xi)^\varepsilon$  can be replaced by  $X^\varepsilon$  (after adjusting the numerical value of  $\varepsilon$ ).

To obtain a basis estimate for  $M_\alpha$ , we use the following generalization of Theorem 1.2 which we prove in Section 14.

**Theorem 3.1.** *Let  $Q \geq 1$  and  $\alpha$  be a complex number in the box  $0 \leq \operatorname{Re}(\alpha) \leq \varepsilon$ ,  $\operatorname{Im}(\alpha) \ll Q^\varepsilon$ , where  $\varepsilon$  is a sufficiently small positive constant. Let further  $\psi$  be a primitive character of conductor  $k$ . Then*

$$(30) \quad \sum_{q \leq Q} \sum_{\substack{\chi \pmod{q} \\ \chi^3 = \chi_0}}^{**} |L(1/2 + \alpha, \chi\psi)|^2 \\ \ll (Qk)^\varepsilon \min \{ Q^{5/3} + Q^{1/2}k^{1/2}, Q^{4/3} + Qk^{1/2}, Q^{11/9} + Q^{7/6}k^{1/2}, Q^{6/5}k^{6/5} \},$$

where the double star indicates that  $\chi$  runs over primitive characters such that  $\chi\psi$  is primitive of conductor  $qk$ .

The above Theorem 3.1 together with the Cauchy-Schwarz inequality implies the basis estimate

$$(31) \quad M_\alpha(w, X, r, \xi, \beta, a) \\ \ll X^\varepsilon \min \{ Q^{4/3} + Q^{3/4}N(r)^{1/4}, Q^{7/6} + QN(r)^{1/4}, Q^{10/9} + Q^{13/12}N(r)^{1/4}, Q^{11/10}N(r)^{3/5} \}.$$

Of course the quality of this estimate is far from the result stated in Theorem 1.1, but we shall apply estimates of this kind for values of the parameters that are small enough for it to be useful.

To continue the estimations we use the approximate functional equation to write  $M_\alpha = M_1 + M_2$ , where

$$(32) \quad M_1 = \sum'_{\substack{(n, r\bar{r}\xi)=1 \\ n \equiv \beta \pmod{9}}} \chi_{nr}(a)w \left( \frac{N(n)}{X} \right) \sum_{m=1}^{\infty} \frac{\chi_{nr}(m)}{m^{\frac{1}{2}+\alpha}} V_\alpha \left( \frac{m}{A_n} \right),$$

and

$$(33) \quad M_2 = \sum'_{\substack{(n, r\bar{r}\xi)=1 \\ n \equiv \beta \pmod{9}}} \chi_{nr}(a)w \left( \frac{N(n)}{X} \right) X_\alpha \epsilon(\chi_{nr}) \sum_{m=1}^{\infty} \frac{\bar{\chi}_{nr}(m)}{m^{\frac{1}{2}-\alpha}} V_{-\alpha} \left( \frac{m}{B} \right).$$

Here  $B$  is a parameter to be chosen later and  $A_n B = N(n)N(r)$ . Let  $A = XN(r)/B$  so that  $A_n = A \frac{N(n)}{X} \asymp A$ .

Our plan is to directly estimate  $M_2$  using properties of sums of cubic Gauss sums, and to estimate  $M_1$  using a complementary combination of the Pólya-Vinogradov inequality and an estimate of the form (31). This method gives us a new estimate for  $M_\alpha$  that we can then use in place of (31), thereby improving our result on  $M_\alpha$ . We can therefore repeat this process indefinitely to optimize the bound obtained by the method. Not only does this allow us to improve the error term in Theorem 1.1, but it also allows us to increase the sizes of the parameters  $a$  and  $r$ , which has applications respectively to “twisting” the moment by larger integers  $a$ , and to studying nonvanishing of cubic characters of the form  $\chi_k$  with  $k \equiv 0$

(mod  $r$ ). It turns out, but it is not *a priori* obvious, that a single use of (31) is sufficient to obtain a power savings in Theorem 1.1. We chose to present this recursive technique since it could be useful in other applications where the initial bound is insufficient to obtain a desired result (indeed, in an early draft of this article, our estimate for  $M_2$  was not sufficiently strong to obtain the asymptotic formula for the first moment with only one use of the recursion, but after a few applications the error term was successfully controlled). Of course, this recursion allows for an improvement of the error term.

This general method of proof is similar to that used by Luo [L], but the estimates we have available for the second moment are much poorer (he had the essentially optimal bound due to Heath-Brown's cubic large sieve (8)). However, the recursive method we use here shows that it is possible to effectively substitute for such an optimal bound on the second moment.

**3.1. Organization of the proof.** In Section 4 we estimate  $M_1$ , while the estimation of  $M_2$  occurs in Section 5. We compute the main term in terms of an explicit Euler product in Section 6, and perform the recursion in Section 7.

#### 4. ESTIMATING $M_1$

We first extract the main term, say  $M_0$ , arising from the  $m$ 's such that  $\chi_n(am)$  is trivial (i.e. takes the value 1 for all  $(n, am) = 1$ ,  $n \equiv \beta \pmod{9}$ ), writing  $M_1 = M_0 + M'_1$  accordingly. It is easy to see that  $\chi_n(am)$  is trivial only if  $am$  is a cube times a power of 3 (here the condition on the power of 3 depends on what is  $\beta \pmod{9}$ ).

Clearly  $M_0 \ll X$ , and in fact with further work one could show  $M_0 \sim cX$  with an explicit constant  $c \neq 0$  (depending on the parameters  $a, \xi, r$ ). In Section 6 we compute  $M_0$  only in the simpler case where many of the parameters are 1.

We have

$$(34) \quad M'_1 = \sum_{m \in \mathbb{N}}'' \frac{\chi_r(am)}{m^{\frac{1}{2} + \alpha}} f\left(\frac{m}{A}\right),$$

where the double prime on the sum over  $m$  means that  $\chi_n(am)$  is nontrivial, and where

$$(35) \quad f(y) = \sum'_{\substack{(n, r\bar{r}\xi)=1 \\ n \equiv \beta \pmod{9}}} \chi_n(am) w\left(\frac{N(n)}{X}\right) V_\alpha\left(y \frac{X}{N(n)}\right).$$

Now we remove the condition that  $n$  has no rational prime divisor by using Möbius inversion (note that one can uniquely express any  $n \in \mathbb{Z}[\omega]$  as  $n = n_1 n_2$ , where  $n_1 \in \mathbb{N}$ , and  $n_2$  has no rational prime divisor), obtaining

$$(36) \quad f(y) = \sum_{\substack{d \in \mathbb{N} \\ (d, 3amr\bar{r}\xi)=1}} \mu_{\mathbb{N}}(d) \sum_{\substack{(n, r\bar{r}\xi)=1 \\ n \equiv \beta \bar{d} \pmod{9}}}^* \chi_n(am) w\left(\frac{N(n)}{X/d^2}\right) V_\alpha\left(y \frac{X/d^2}{N(n)}\right),$$

where now the  $*$  indicates that  $nd$  is squarefree (viewed as an element of  $\mathbb{Z}[\omega]$ ). Here  $\mu_{\mathbb{N}}$  is the usual Möbius function defined on  $\mathbb{N}$ . Since  $d$  is automatically squarefree (as an element of  $\mathbb{Z}[\omega]$ ),  $nd$  being squarefree simply means that  $n$  is squarefree and  $(n, d) = 1$ . Now use Möbius again (writing  $\mu_\omega$  for the Möbius function on  $\mathbb{Z}[\omega]$ ) to detect the condition that  $n$  is



squarefree, getting

$$(37) \quad f(y) = \sum_{\substack{d \in \mathbb{N} \\ (d, 3amr\bar{r}\xi)=1}} \mu_{\mathbb{N}}(d) \sum_{\substack{(l, dr\bar{r}\xi)=1 \\ l \equiv 1 \pmod{3}}} \mu_{\omega}(l) \\ \sum_{\substack{(n, dr\bar{r}\xi)=1 \\ n \equiv \beta dl^2 \pmod{9}}} \chi_{nl^2}(am)w \left( \frac{N(n)}{X/(dN(l))^2} \right) V_{\alpha} \left( y \frac{X/(dN(l))^2}{N(n)} \right).$$

Here we changed variables via  $n = l^2 n'$ , and we fixed  $l$  up to a unit by the condition  $l \equiv 1 \pmod{3}$  (which then determined  $n'$  which is  $\equiv \pm 1 \pmod{3}$ ).

Now break up the sums over  $d$  and  $l$  into dyadic segments of the form  $D < d \leq 2D$  and  $Y < N(l) \leq 2Y$ , writing  $f_{D,Y}(y)$  for the contribution to  $f(y)$ , and writing  $M_1(D, Y)$  for the contribution to  $M'_1$  of such  $d$  and  $l$ . We will treat large and small values of  $DY$  separately.

**4.1. Small  $DY$ .** In this section we prove

**Lemma 4.1.** *We have*

$$(38) \quad M_1(D, Y) \ll DY A^{3/2} N(a)^{1/2} X^{\varepsilon}.$$

*Proof.* Using (37) and (34), we have

$$(39) \quad M_1(D, Y) \ll \sum_{m \in \mathbb{N}}'' \frac{1}{m^{\frac{1}{2} + \alpha}} \sum_{\substack{D < d \leq 2D \\ (d, 3amr\bar{r}\xi)=1}} \sum_{\substack{Y < N(l) \leq 2Y \\ (l, dr\bar{r}\xi)=1 \\ l \equiv 1 \pmod{3}}} \left| \sum_{\substack{(n, dr\bar{r}\xi)=1 \\ n \equiv \beta \pmod{9}}} \chi_{nl^2}(am)w \left( \frac{N(n)}{X/(dN(l))^2} \right) V_{\alpha} \left( \frac{m X/(dN(l))^2}{A N(n)} \right) \right|$$

The inner sum over  $n$  is bounded by  $X^{\varepsilon} N(am)^{1/2}$  by using the Pólya-Vinogradov inequality (after using cubic reciprocity, Möbius inversion, and the ray class characters to detect the congruence  $\pmod{9}$ ). We use this estimate for  $m \ll AX^{\varepsilon}$ , and otherwise use the trivial bound on the sum over  $n$  and rapid decay of  $V_{\alpha}$  to find that the contribution from  $m \gg AX^{\varepsilon}$  is negligible.

$$(40) \quad M_1(D, Y) \ll DY A^{3/2} N(a)^{1/2} X^{\varepsilon},$$

as desired.  $\square$

**4.2. Large  $DY$ .** For large  $DY$  we wish to use the method of going back to primitive characters, which we can trace to the paper of Iwaniec [I], Section 6. The point is to obtain an expression similar to (29), but with the lengths of summation shortened. One new feature here is that we express the contribution from large  $DY$  in terms of the average value of the  $L$ -functions (rather than the absolute value), which is necessary because our methods do not treat the absolute values of the  $L$ -functions (which essentially would require going to the second moment). The notation and details become rather messy but the underlying ideas are quite simple.

Our result of this section is

**Lemma 4.2.** *Suppose that for nonnegative real numbers  $\gamma, \delta, \nu$  satisfying  $2\gamma - \delta \geq 1$ ,  $\delta, \nu \geq 0$ ,  $\delta \geq \nu - 1/4$  the following holds*

$$(41) \quad M_\alpha(w, X, r, \xi, \beta, a) \ll X^\varepsilon (X + X^\gamma N(r)^\delta N(a)^\nu).$$

Then

$$(42) \quad M_1(D, Y) \ll X^\varepsilon \left( \frac{X}{DY} + A^{1/2} X^{1/2} + \frac{X^\gamma N(r)^\delta N(a)^\nu}{(DY)^{2\gamma-\delta-1}} \right).$$

Note that the case  $\gamma = 11/10$ ,  $\delta = 3/5$ ,  $\nu = 0$  is provided by (31).

*Proof.* If  $DY$  is large, we go back to  $n$  being squarefree and having no rational prime divisor, by successively writing  $n = en_1$  where  $e$  is rational and  $n_1$  has no rational prime divisor, and then  $n_1 = k^2 n'$  where  $n'$  is squarefree, both  $k$  and  $n'$  have no rational prime divisors,  $(n', \bar{k}) = 1$ , and  $k \equiv 1 \pmod{3}$ . Thus we obtain

$$(43) \quad f_{D,Y}(y) = \sum_{\substack{d \in \mathbb{N} \\ (d, 3amr\bar{r}\xi)=1 \\ D < d \leq 2D}} \mu_{\mathbb{N}}(d) \sum_{\substack{(l, dr\bar{r}\xi)=1 \\ l \equiv 1 \pmod{3} \\ Y < N(l) \leq 2Y}} \mu_\omega(l) \sum_{\substack{k \text{ no } \mathbb{N} \text{ divisors} \\ (k, dr\bar{r}\xi)=1 \\ k \equiv 1 \pmod{3}}} \sum_{\substack{e \in \mathbb{N} \\ (e, 3adm\bar{r}\bar{r}\xi)=1}} \chi_{n'(kl)^2}(am)w \left( \frac{N(n')}{X/(deN(kl))^2} \right) V_\alpha \left( y \frac{X/(deN(kl))^2}{N(n')} \right) \\ \sum'_{\substack{(n', dr\bar{r}\xi\bar{k})=1 \\ n' \equiv \beta dek^2 l^2 \pmod{9}}} \left( \frac{N(n')}{X/(deN(kl))^2} \right) V_\alpha \left( y \frac{X/(deN(kl))^2}{N(n')} \right).$$

Then let  $de = c$  and  $kl = j$  be new variables to obtain

$$(44) \quad f_{D,Y}(y) = \sum_{\substack{c \in \mathbb{N} \\ (c, 3amr\bar{r}\xi)=1 \\ c > D}} \left( \sum_{\substack{d|c \\ (d, c/d)=1 \\ D < d \leq 2D}} \mu_{\mathbb{N}}(d) \right) \sum_{\substack{(j, dr\bar{r}\xi)=1 \\ j \equiv 1 \pmod{3} \\ N(j) > Y}} \left( \sum_{\substack{l|j \\ l \equiv 1 \pmod{3} \\ Y < N(l) \leq 2Y \\ j/l \text{ no } \mathbb{N} \text{ divisors}}} \mu_\omega(l) \right) \\ \sum'_{\substack{(n', dr\bar{r}\xi(j/l))=1 \\ n' \equiv \beta cj^2 \pmod{9}}} \chi_{n'j^2}(am)w \left( \frac{N(n')}{X/(cN(j))^2} \right) V_\alpha \left( y \frac{X/(cN(j))^2}{N(n')} \right).$$

Inserting this expression into (34) and extending  $m$  back to all positive integers gives (remembering that  $M_1(D, Y)$  is the contribution to  $M'_1$  of the terms with  $D < d \leq 2D$ ,  $Y < N(l) \leq 2Y$ )

$$(45) \quad M_1(D, Y) = \sum_{\substack{c \in \mathbb{N} \\ (c, 3ar\bar{r}\xi)=1 \\ c > D}} \left( \sum_{\substack{d|c \\ (d, c/d)=1 \\ D < d \leq 2D}} \mu_{\mathbb{N}}(d) \right) \sum_{\substack{(j, dr\bar{r}\xi)=1 \\ j \equiv 1 \pmod{3} \\ N(j) > Y}} \left( \sum_{\substack{l|j \\ l \equiv 1 \pmod{3} \\ Y < N(l) \leq 2Y \\ j/l \text{ no } \mathbb{N} \text{ divisors}}} \mu_\omega(l) \right) N + O \left( \frac{X^{1+\varepsilon}}{aDY} \right),$$

where

$$(46) \quad N = \sum'_{\substack{(n', dr\bar{r}\xi(j/\bar{l}))=1 \\ n' \equiv \beta c j^2 \pmod{9}}} \sum_{(m,c)=1} \frac{\chi_{n' r j^2}(am)}{m^{\frac{1}{2}+\alpha}} w\left(\frac{N(n')}{X'}\right) V_\alpha\left(\frac{m}{A} \frac{X'}{N(n')}\right),$$

and  $X' = X/(cN(j))^2$ . Here the error term comes from putting back the terms where  $m$  is  $a^2$  times a cube.

We now relax some of the coprimality conditions, obtaining

$$(47) \quad M_1(D, Y) \ll X^\varepsilon \sum_{c \in \mathbb{N}, c > D} \sum_{\substack{(j, r\bar{r})=1 \\ N(j) > Y}} \left( \sum_{\substack{l|j, l \equiv 1 \pmod{3} \\ Y < N(l) \leq 2Y \\ j/l \text{ no } \mathbb{N} \text{ divisors}}} |\mu_\omega(l)| \right) |N| + O\left(\frac{X^{1+\varepsilon}}{aDY}\right).$$

To get to primitive characters, we write  $j = j_0 j_1 j_2^2 j_3^3$ , where  $j_0 \in \mathbb{N}$ ,  $j/j_0$  has no rational prime divisor, and  $j_1 j_2$  is squarefree. Then  $\chi_{n' r j^2}(am) = \chi_{n' r \bar{j}_1 \bar{j}_2}(am) \chi_{0, j_0 j_3}(am)$ , where  $\chi_{0,k}$  is the principal character of modulus  $N(k)$ . Note that  $(\bar{j}_1, j_2) = 1$  since  $j_1 j_2$  has no rational prime divisor. Then write  $n' = n_1 n'_1 n_2 n'_2 n''$ , where  $n_1 | j_1$ ,  $n'_1 | \bar{j}_1$ ,  $n_2 | j_2$ ,  $n'_2 | \bar{j}_2$ , and  $(n'', j_1 \bar{j}_1 j_2 \bar{j}_2) = 1$ . Similarly write  $\bar{j}_1 = \bar{n}_1 n'_1 j'_1$  and  $j_2 = n_2 \bar{n}_2 j'_2$ , noting that  $(n_1, n'_1) = 1$ , and similarly for  $n_2$ , which follow since  $j_1$  and  $j_2$  have no rational prime divisor. Then  $\chi_{n' r \bar{j}_1 j_2} = \chi_{n'' r \bar{n}_1 \bar{n}_2 j'_1 j'_2} \chi_{0, n_1 n'_2}$ . Notice that  $n'' r \bar{n}_1 \bar{n}_2 j'_1 j'_2$  has no rational prime divisor and is square-free so that this character is primitive.

Hence

$$(48) \quad N = \sum_{\substack{n_1, n'_1, n_2, n'_2 \\ \dots}} \sum'_{(n'', dr\bar{r}j_1 \bar{j}_1 j_2 \bar{j}_2 \xi(\frac{\bar{l}}{l}))=1} \sum_{(m,c)=1} m^{-\frac{1}{2}-\alpha} \chi_{n'' r \bar{n}_1 \bar{n}_2 j'_1 j'_2}(am) \chi_{0, j_0 j_3 n_1 n'_2}(am) w\left(\frac{N(n')}{X'}\right) V_\alpha\left(\frac{m}{A} \frac{X'}{N(n')}\right).$$

Here the dots on the sum over  $n_1, n'_1$ , etc. indicate various coprimality relations as well as the fact that  $n_1 \bar{n}'_1 | j_1$  and  $n_2 \bar{n}'_2 | j_2$ , and that  $n'' \equiv \beta' \pmod{9}$  for some  $\beta' \equiv 1 \pmod{3}$  depending on other parameters.

We reduce to understanding

$$(49) \quad N' = \sum'_{\substack{(n'', dr\bar{r}j_1 \bar{j}_1 j_2 \bar{j}_2 \xi(\bar{j}/\bar{l}))=1 \\ n'' \equiv \beta' \pmod{9}}} \sum_{(m, c j_0 j_3 n_1 n'_2)=1} \frac{\chi_{n'' r \bar{n}_1 \bar{n}_2 j'_1 j'_2}(am)}{m^{\frac{1}{2}+\alpha}} w\left(\frac{N(n')}{X'}\right) V_\alpha\left(\frac{m}{A} \frac{X'}{N(n')}\right).$$

Now use the Mellin transform of  $V_\alpha$  to get

$$(50) \quad N' = \sum'_{\substack{(n'', dr\bar{r}j_1 \bar{j}_1 j_2 \bar{j}_2 \xi(\bar{j}/\bar{l}))=1 \\ n'' \equiv \beta' \pmod{9}}} \chi_{n'' r \bar{n}_1 \bar{n}_2 j'_1 j'_2}(a) \frac{1}{2\pi i} \int_{(1)} A^s L(\tfrac{1}{2} + \alpha + s, \chi_{n'' r \bar{n}_1 \bar{n}_2 j'_1 j'_2} \chi_{0, c j_0 j_3 n_1 n'_2}) w_s\left(\frac{N(n')}{X'}\right) \frac{G(s)}{s} g(s) ds,$$

where  $w_s(x) = x^s w(x)$ . Next we move the line of integration to  $\varepsilon$  without crossing any poles unless  $n'' = r = n'_1 = n_2 = j'_1 = j'_2 = 1$ . In this case,  $n' = n_1 n'_2$  and  $j_1 = n_1$ ,  $j_2 = \bar{n}'_2$ ,

so that given  $j$ , the sum over  $n'$  is bounded. Since  $N(j)^2 \ll c^{-2}X$ , we easily see that the contribution of this pole to  $M_1(D, Y)$  is

$$(51) \quad \ll A^{\frac{1}{2}} X^{\frac{1}{2}+\varepsilon}.$$

On the new line of integration we use that the integrand has exponential decay so that we may truncate the integral to lie in the region  $\text{Im}(s) \ll X^\varepsilon$ , at no cost. Writing  $N''$  for this new integral, and removing the principal character  $\chi_{0, cj_0 j_3 n_1 n'_2}$  gives

$$(52) \quad N'' = \sum_{\substack{b \in \mathbb{N} \\ b|N(cj_0 j_3 n_1 n'_2)}} \frac{\mu_{\mathbb{N}}(b)}{b^{\frac{1}{2}+\alpha}} \sum'_{\substack{(n'', d\bar{r}\bar{j}_1\bar{j}_1\bar{j}_2\bar{j}_2\xi(\bar{j}/l))=1 \\ n'' \equiv \beta' \pmod{9}}} \chi_{n'' r \bar{n}'_1 \bar{n}'_2 j'_1 j'_2}(ab) \\ \frac{1}{2\pi i} \int_{\substack{s=\varepsilon+it \\ |t| \ll X^\varepsilon}} A^s b^{-s} L\left(\frac{1}{2} + \alpha + s, \chi_{n'' r \bar{n}'_1 \bar{n}'_2 j'_1 j'_2}\right) w_s \left( \frac{N(n'')}{X''} \right) \frac{G(s)}{s} g(s) ds,$$

where  $X'' = X'/N(n_1 n'_1 n_2 n'_2)$

Hence we have

$$(53) \quad N'' \ll X^\varepsilon \sum_{b|N(cj_0 j_3 n_1 n'_2)} \frac{|\mu_{\mathbb{N}}(b)|}{b^{\frac{1}{2}}} \sup_{\substack{s=\varepsilon+it \\ |t| \ll X^\varepsilon}} M_{\alpha+s}(w, X'', r j'_1 j'_2 \bar{n}'_1 \bar{n}'_2, \xi d(\bar{j}/l) \frac{j_1 j_2}{j'_1 j'_2 \bar{n}'_1 \bar{n}'_2}; \chi(ab)).$$

Using the hypothesis (41), we obtain

$$(54) \quad N'' \ll X^\varepsilon \sum_{b|N(cj_0 j_3 n_1 n'_2)} \frac{|\mu_{\mathbb{N}}(b)|}{b^{\frac{1}{2}}} (X'' + (X'')^\gamma N(r j'_1 j'_2 \bar{n}'_1 \bar{n}'_2)^\delta N(ab)^\nu)$$

$$(55) \quad \ll X^\varepsilon \left( X'' + (X'')^\gamma N(r j'_1 j'_2 \bar{n}'_1 \bar{n}'_2)^\delta N(a)^\nu N(cj_0 j_3 n_1 n'_2)^{\max\{0, \nu - \frac{1}{4}\}} \right)$$

$$(56) \quad \ll X^\varepsilon (X'' + (X'')^\gamma N(c r j_0 j_1 j_2 j_3)^\delta N(a)^\nu).$$

Clearly the contribution of  $N''$  to  $N$  is

$$(57) \quad \ll X^\varepsilon (X' + X'^\gamma N(c r j)^\delta N(a)^\nu),$$

because the the sum over  $n_1, n'_1, n_2, n'_2$  is over some restricted set of divisors of  $j$ , of which there are  $\ll X^\varepsilon$ , and because  $X'' \ll X'$ .

Gathering these estimates, we get

$$(58) \quad M_1(D, Y) \ll \frac{X}{DY} + A^{1/2} X^{1/2+\varepsilon} + X^\varepsilon \sum_{c>D} \sum_{N(j)>Y} \left( \frac{X}{c^2 N(j)^2} + \left( \frac{X}{c^2 N(j)^2} \right)^\gamma N(c j r)^\delta N(a)^\nu \right)$$

Thus

$$(59) \quad M_1(D, Y) \ll X^\varepsilon \left( \frac{X}{DY} + A^{1/2} X^{1/2} + \frac{X^\gamma N(r)^\delta N(a)^\nu}{(DY)^{2\gamma-\delta-1}} \right),$$

as claimed. □

**4.3. Combining the estimates on  $M_1$ .** In this section we show

**Lemma 4.3.** *Assuming that (41) holds, then*

$$(60) \quad M'_1 \ll X^\varepsilon \left( X^{\frac{\gamma}{2\gamma-\delta}} A^{\frac{3(2\gamma-\delta-1)}{2(2\gamma-\delta)}} N(r)^{\frac{\delta}{2\gamma-\delta}} N(a)^{\frac{\nu-\frac{1}{2}}{2\gamma-\delta}+\frac{1}{2}} + X^{1/2} A^{3/4} N(a)^{1/4} \right)$$

*holds.*

*Proof.* To obtain this estimate for  $M_1'$  we need to sum over  $D$ ,  $Y$  over dyadic segments. We use Lemma 4.2 to cover the case where  $DY$  is large, and Lemma 4.1 to cover the complementary set. We obtain  $\ll \log^2 X$  such dyadic segments, and it turns out that for the parameters  $\gamma, \delta, \nu$  that derive from the large sieve, the worst bound is when

$$(61) \quad \frac{X^\gamma N(r)^\delta N(a)^\nu}{(DY)^{2\gamma-\delta-1}} \asymp DY A^{3/2} N(a)^{\frac{1}{2}}.$$

That is,

$$(62) \quad (DY)^{2\gamma-\delta} \asymp A^{-3/2} X^\gamma N(r)^\delta N(a)^{\nu-\frac{1}{2}}.$$

It is perhaps simplest to also compare  $DY A^{3/2} N(a)^{1/2}$  and  $(DY)^{-1} X$  since here the worst situation is for  $DY \asymp X^{1/2} A^{-3/4} N(a)^{-1/4}$ , which leads to a bound of the size  $X^{1/2} A^{3/4} N(a)^{1/4}$ , which dwarfs the remaining term  $X^{1/2} A^{1/2}$ .  $\square$

Remark. If the term  $X^\gamma N(r)^\delta N(a)^\nu$  in our hypothesis (41) is replaced by a sum of several terms of the form  $X^{\gamma_i} N(r)^{\delta_i} N(a)^{\nu_i}$ , then we obtain the same bound as (60) with the term depending on  $\gamma, \delta, \nu$  replaced by the corresponding sum of terms depending on  $\gamma_i, \delta_i, \nu_i$ .

## 5. ESTIMATING $M_2$

Recall

$$(63) \quad M_2 = \sum'_{\substack{(n, r\bar{r}\xi)=1 \\ n \equiv \beta \pmod{9}}} \chi_{nr}(a) w \left( \frac{N(n)}{X} \right) X_\alpha \epsilon(\chi_{nr}) \sum_{m=1}^{\infty} \frac{\bar{\chi}_{nr}(m)}{m^{\frac{1}{2}-\alpha}} V_{-\alpha} \left( \frac{m}{B} \right).$$

In this section we show

**Lemma 5.1.** *We have*

$$(64) \quad M_2 \ll X^\epsilon \left( B^{1/6} X^{5/6} + X^{1/2} B N(a)^{1/2} \right).$$

The cancellation here comes from the sum of cubic Gauss sums, which follows from the work of Patterson showing that these cubic Gauss sums appear as Fourier coefficients of metaplectic Eisenstein series [P1].

*Proof.* By a calculation in Section 2.3, we have  $\epsilon(\chi_n) = i^{-1} g(n) N(n)^{-\frac{1}{2}}$ , so that

$$(65) \quad M_2 = i^{-1} X^{-\alpha} \sum_{m=1}^{\infty} \frac{\bar{\chi}_r(a^2 m) \tilde{g}(r)}{m^{\frac{1}{2}-\alpha}} V_{-\alpha} \left( \frac{m}{B} \right) \sum'_{\substack{(n, r\bar{r}\xi)=1 \\ n \equiv \beta \pmod{9}}} \frac{\bar{\chi}_n(a^2 m) g(n)}{N(n)^{\frac{1}{2}}} w_\alpha \left( \frac{N(n)}{X} \right),$$

where

$$(66) \quad w_\alpha(x) = w(x) (x/\pi)^{-\alpha} \frac{\Gamma\left(\frac{\frac{3}{2}-\alpha}{2}\right)}{\Gamma\left(\frac{\frac{3}{2}+\alpha}{2}\right)}.$$

Next we remove the condition that  $n$  has no rational prime divisor using Möbius, getting

$$(67) \quad M_2 = i^{-1} X^{-\alpha} \sum_{d \in \mathbb{N}, (d, 3a)=1} \mu_{\mathbb{N}}(d) \tilde{g}(d) \sum_{(m, d)=1} \frac{\bar{\chi}_r(a^2 m) \tilde{g}(r)}{m^{\frac{1}{2}-\alpha}} V_{-\alpha} \left( \frac{m}{B} \right) \\ \sum_{\substack{(n, r\bar{r}d\xi)=1 \\ n \equiv \bar{d}\beta \pmod{9}}} \frac{\bar{\chi}_n(a^2 m) g(n)}{N(n)^{\frac{1}{2}}} w_{\alpha} \left( \frac{N(n)}{X/d^2} \right).$$

Here we could drop the condition that  $n$  is squarefree since  $g(n) = 0$  otherwise. Using the ray class characters  $\lambda \pmod{9}$  gives that  $M_2$  is a linear combination of terms of the form

$$(68) \quad M_2(\lambda) = i^{-1} X^{-\alpha} \sum_{d \in \mathbb{N}, (d, a)=1} \mu_{\mathbb{N}}(d) \bar{\lambda}(d) \tilde{g}(d) \sum_{(m, d)=1} \frac{\bar{\chi}_r(a^2 m) \tilde{g}(r)}{m^{\frac{1}{2}-\alpha}} V_{-\alpha} \left( \frac{m}{B} \right) \\ \sum_{\substack{(n, r\bar{r}d\xi)=1 \\ n \equiv 1 \pmod{3}}} \frac{\lambda(n) \bar{\chi}_n(a^2 m) g(n)}{N(n)^{\frac{1}{2}}} w_{\alpha} \left( \frac{N(n)}{X/d^2} \right).$$

Now introduce the Mellin transform of  $w_{\alpha}$  to get

$$(69) \quad M_2(\lambda) = i^{-1} X^{-\alpha} \sum_{d \in \mathbb{N}, (d, a)=1} \mu_{\mathbb{N}}(d) \bar{\lambda}(d) \tilde{g}(d) \sum_{(m, d)=1} \frac{\bar{\chi}_r(a^2 m) \tilde{g}(r)}{m^{\frac{1}{2}-\alpha}} V_{-\alpha} \left( \frac{m}{B} \right) \\ \frac{1}{2\pi i} \int_{(2)} \widetilde{w}_{\alpha}(s) \left( \frac{X}{d^2} \right)^s h\left(\frac{1}{2} + s, a^2 m, r\bar{r}d\xi\right) ds,$$

where

$$(70) \quad h(s, e, f) = \sum_{(n, f)=1} \frac{\lambda(n) \overline{\chi_e(n)} g(n)}{N(n)^s}.$$

Clearly the series defining  $h(s, e, f)$  converges absolutely and uniformly on any region  $\operatorname{Re}(s) \geq \frac{3}{2} + \delta > \frac{3}{2}$ .

The required analytic properties of  $h$  are summarized with

**Lemma 5.2.**  *$h(s, e, f)$  extends to a meromorphic function on  $\mathbb{C}$  with simple poles at  $s = 4/3$ ,  $s = 2/3$  only. Furthermore, letting  $\sigma_1 = \frac{3}{2} + \varepsilon$ , and  $\sigma_1 \geq \sigma \geq \sigma_1 - \frac{1}{2}$ ,  $|s - \frac{4}{3}| > \frac{1}{12}$ , we have*

$$(71) \quad h(s, e, f) \ll N(f)^{\varepsilon} N(e)^{\frac{1}{2}(\sigma_1 - \sigma)} (1 + t^2)^{\sigma_1 - \sigma}.$$

The residue satisfies

$$(72) \quad \operatorname{res}_{s=4/3} h(s, e, f) \ll N(e f)^{\varepsilon} N(e)^{-1/6}.$$

Before proving Lemma 5.2 we complete our estimation of  $M_2$ . We move the line of integration to  $\operatorname{Re}(s) = \frac{1}{2} + \varepsilon$ , crossing a pole at  $s = 5/6$ , which contributes

$$(73) \quad M_{2, \text{residue}} \ll X^{\varepsilon} \sum_{d \in \mathbb{N}} \sum_m m^{-1/2} |V(m/B)| \left( \frac{X}{d^2} \right)^{5/6} N(a^2 m)^{-1/6} \ll X^{\varepsilon} B^{1/6} X^{5/6}.$$

The main contribution comes from the new line of integration, which gives

$$(74) \quad M_{2,\text{integral}} \ll X^\varepsilon \sum_{d \in \mathbb{N}} \sum_{m=1}^{\infty} m^{-1/2} |V(m/B)| \left( \frac{X}{d^2} \right)^{1/2+\varepsilon} N(a^2 m)^{\frac{1}{4}+\varepsilon} \ll X^\varepsilon X^{1/2} B N(a)^{1/2}.$$

This completes the proof of Lemma 4.3.  $\square$

*Proof of Lemma 5.2.* In this section we use the convention that all sums over elements of  $\mathbb{Z}[\omega]$  are restricted to elements  $\equiv 1 \pmod{3}$ . We have

$$(75) \quad h(s, e, f) = \sum_{(n, ef)=1} \frac{\lambda(n)g(e, n)}{N(n)^s}.$$

Our goal is to manipulate the expression to eventually remove the coprimality condition. Obviously we may assume  $f$  is squarefree, and that  $(e, f) = 1$ . Using Möbius to remove the condition  $(n, f) = 1$  gives

$$(76) \quad h(s, e, f) = \sum_{r|f} \frac{\mu_\omega(r)\lambda(r)}{N(r)^s} \sum_{(n, e)=1} \frac{\lambda(n)g(e, rn)}{N(n)^s}.$$

Notice that if  $\pi|r$  then  $\pi \nmid e$  so if in addition  $\pi|n$  then  $g(e, rn) = 0$ , using (27). Thus we may assume  $(n, r) = 1$ , in which case we have  $g(e, rn) = g(re, n)g(e, r)$  by (25), and hence

$$(77) \quad h(s, e, f) = \sum_{r|f} \frac{\mu_\omega(r)\lambda(r)g(e, r)}{N(r)^s} h_1(s, er),$$

where

$$(78) \quad h_1(s, m) = \sum_{(n, m)=1} \frac{\lambda(n)g(m, n)}{N(n)^s}.$$

Now write  $m = m_1 m_2^2 m_3^3$ , where  $m_1 m_2$  is squarefree. Then from (23) it follows that  $g(m, n) = g(m_1 m_2^2, n)$  provided  $(n, m_3) = 1$ . So we get

$$(79) \quad h_1(s, m) = \sum_{(n, m_1 m_2 m_3^*)=1} \frac{\lambda(n)g(m_1 m_2^2, n)}{N(n)^s},$$

where  $m_3^* = \prod_{\pi|m_3} \pi$ . Note that  $h_1(s, m_1 m_2^2 m_3^3) = h(s, m_1 m_2^2, m_3^*)$ , so using (77) we get

$$(80) \quad h_1(s, m) = h(s, m_1 m_2^2, m_3^*) = \sum_{t|m_3^*} \frac{\mu_\omega(t)\lambda(t)g(m_1 m_2^2, t)}{N(t)^s} h_1(s, m_1 m_2^2).$$

Now we manipulate  $h_1(s, m_1 m_2^2)$ ; actually we make a slight generalization as follows. Let  $ab^2 \in \mathbb{Z}[\omega]$  and let  $\pi$  be prime such that  $(ab, \pi) = 1$ . Then

$$(81) \quad h_2(s, a\pi^2, b^2) := \sum_{(n, a\pi)=1} \frac{\lambda(n)g(ab^2\pi^2, n)}{N(n)^s} = \sum_{(n, a)=1} \frac{\lambda(n)g(ab^2\pi^2, n)}{N(n)^s} - \sum_{(n, a)=1, \pi|n} \frac{\lambda(n)g(ab^2\pi^2, n)}{N(n)^s}.$$

Writing in the latter sum  $n = \pi^j n'$ , where  $(n', \pi) = 1$ , then we have  $g(ab^2\pi^2, \pi^j n') = g(\pi^{j+2}ab^2, n')g(ab^2\pi^2, \pi^j)$  by (25). Using (23) we get  $g(ab^2\pi^2, \pi^j) = \overline{(ab^2/\pi^j)}_3 g(\pi^2, \pi^j)$ , which

is nonzero if and only if  $j = 3$ , from (26). Thus we get  $g(ab^2\pi^2, \pi^3n') = -N(\pi^2)g(ab^2\pi^2, n')$ . In summary, we have shown

$$(82) \quad h_2(s, a\pi^2, b^2) = \sum_{(n,a)=1} \frac{\lambda(n)g(ab^2\pi^2, n)}{N(n)^s} + \lambda(\pi)^3 N(\pi)^{2-3s} h_2(s, a\pi^2, b^2),$$

which when rearranged states

$$(83) \quad \sum_{(n,a\pi)=1} \frac{\lambda(n)g(ab^2\pi^2, n)}{N(n)^s} = (1 - \lambda(\pi^3)N(\pi)^{2-3s})^{-1} \sum_{(n,a)=1} \frac{\lambda(n)g(ab^2\pi^2, n)}{N(n)^s}.$$

An induction argument on the number of prime divisors of  $b$  gives

$$(84) \quad h_1(s, m_1 m_2^2) = \prod_{\pi|m_2} (1 - \lambda(\pi)^3 N(\pi)^{2-3s})^{-1} \sum_{(n,m_1)=1} \frac{\lambda(n)g(m_1 m_2^2, n)}{N(n)^s}.$$

Then we have

$$(85) \quad \sum_{(n,m_1)=1} \frac{\lambda(n)g(m_1 m_2^2, n)}{N(n)^s} = \widetilde{\psi}_{m_1}(m_1 m_2^2, s, 0),$$

where  $\widetilde{\psi}$  is a slight generalization of definition on p.124 of [H-BP]. The only difference is that we have the ray class character  $\lambda(n)$  appearing. According to Lemma 3 of [H-BP], we have (for  $\lambda = 1$ ) that

$$(86) \quad \widetilde{\psi}_{m_1}(m_1 m_2^2, s, 0) = \prod_{\pi|m_1} (1 - N(\pi)^{2-3s})^{-1} \sum_{d|m_1} \mu_\omega(d) N(d)^{1-2s} \overline{g(m_1 m_2^2/d, d)} \psi(m_1 m_2^2/d, s, 0).$$

An inspection of the proof shows that (86) continues to hold for our generalized  $\psi$  multiplied by  $\lambda(n)$  since the ray class character is completely multiplicative.

The meromorphic continuation of  $\psi$  (without  $\lambda$ ) is given by Theorem 6.1 of [P1], and its residue at  $s = 4/3$  is given by Proposition 8.4 of [LP], while its growth in vertical strips is given by Lemma 4 of [H-BP]. The necessary generalizations to include  $\lambda$  have been carried out in [P2], e.g. We have the bound

$$(87) \quad \psi(r, s, 0) \ll N(r)^{\frac{1}{2}(\sigma_1 - \sigma)} (1 + t^2)^{\sigma_1 - \sigma},$$

where  $\sigma_1 = \frac{3}{2} + \varepsilon$ , and  $\sigma_1 \geq \sigma \geq \sigma_1 - \frac{1}{2}$ ,  $|s - \frac{4}{3}| > \frac{1}{12}$ . We work our way backwards, obtaining first that

$$(88) \quad \widetilde{\psi}_{m_1}(m_1 m_2^2, s, 0) \ll N(m_1 m_2^2)^{\frac{1}{2}(\sigma_1 - \sigma)} (1 + t^2)^{\sigma_1 - \sigma}.$$

We observe that  $h_1(s, m_1 m_2^2)$  satisfies the same bound, and that

$$(89) \quad h_1(s, m) \ll N(m)^{\frac{1}{2}(\sigma_1 - \sigma)} (1 + t^2)^{\sigma_1 - \sigma}.$$

Finally we obtain

$$(90) \quad h(s, e, f) \ll N(f)^\varepsilon N(e)^{\frac{1}{2}(\sigma_1 - \sigma)} (1 + t^2)^{\sigma_1 - \sigma}$$

As for the residue, we use Theorems 9.1 and 8.1 of [P1], to get that  $\psi(m_1 m_2^2/d, s, 0)$  has residue of size  $\ll N(m_1/d)^{-1/6}$ . We easily conclude that

$$(91) \quad \operatorname{res}_{s=4/3} h(s, e, f) \ll N(e f)^\varepsilon N(e)^{-1/6},$$

which finishes the proof of Lemma 5.2. □



6. COMPUTING  $M_0$ 

Recall from (32) (setting  $a = r = \xi = 1$ ) that  $M_0$  is the sum over  $m$  such that  $\chi_n(m) = 1$  for all  $(n, m) = 1$ ,  $n \equiv \beta \pmod{9}$ . Note that we can write such  $m$  uniquely as  $3^t$  times a cube, where  $t = 0, 1, 2$ , and  $\chi_\beta(3)^t = 1$  (so in particular  $t = 1, 2$  occur here only if  $\chi_\beta(3) = 1$ ). Thus

$$(92) \quad M_0 = \sum_{\substack{t \in \{0,1,2\} \\ \chi_\beta(3)^t = 1}} \sum'_{n \equiv \beta \pmod{9}} w\left(\frac{N(n)}{X}\right) \sum_{m \in \mathbb{N}, (m, n\bar{n})=1} \frac{1}{3^{t(\frac{1}{2}+\alpha)} m^{\frac{3}{2}+3\alpha}} V_\alpha\left(\frac{3^t m^3}{A_n}\right).$$

We will treat the case  $\beta \equiv 1 \pmod{9}$ ,  $t = 0$  for notational simplicity. Each case leads to a main term of the form  $c_{t,\beta} X \widehat{w}(0)$  where  $c_{t,\beta} > 0$ .

Using the Mellin transform of  $V$ , we have

$$(93) \quad M_0 = \sum'_{n \equiv 1 \pmod{9}} w\left(\frac{N(n)}{X}\right) \frac{1}{2\pi i} \int_{(\varepsilon)} \left(\frac{AN(n)}{X}\right)^s \frac{G(s)}{s} g_\alpha(s) \zeta_{n\bar{n}}\left(\frac{3}{2} + 3\alpha + 3s\right) ds,$$

where

$$(94) \quad \zeta_{n\bar{n}}(s) = \zeta(s) \sum_{a|n\bar{n}, a \in \mathbb{N}} \mu_{\mathbb{N}}(a) a^{-s}.$$

Thus we have

$$(95) \quad M_0 = \sum'_{n \equiv 1 \pmod{9}} \sum_{a|N(n), a \in \mathbb{N}} \frac{\mu_{\mathbb{N}}(a)}{a^{\frac{3}{2}+3\alpha}} w\left(\frac{N(n)}{X}\right) \frac{1}{2\pi i} \int_{(\varepsilon)} \left(\frac{AN(n)}{Xa^3}\right)^s \frac{G(s)}{s} g_\alpha(s) \zeta\left(\frac{3}{2} + 3\alpha + 3s\right) ds.$$

We move the line of integration to  $-\frac{1}{2} + \varepsilon$ , crossing poles at  $s = 0$  and at  $s = -\frac{1}{6} + \alpha$ . On the new line we use the trivial bound to obtain an error of size

$$(96) \quad A^{-\frac{1}{2}} X^{1+\varepsilon}.$$

We choose  $G(s)$  to vanish at  $s = -\frac{1}{6} + \alpha$  so that this term is illusory. The pole at  $s = 0$  gives

$$(97) \quad M_{\text{res}} = \zeta\left(\frac{3}{2} + 3\alpha\right) \sum'_{n \equiv 1 \pmod{9}} \sum_{a|N(n)} \frac{\mu_{\mathbb{N}}(a)}{a^{\frac{3}{2}+3\alpha}} w\left(\frac{N(n)}{X}\right).$$

At this point we use the ray class characters to detect the condition  $n \equiv 1 \pmod{9}$ , getting

$$(98) \quad M_{\text{res}} = \zeta\left(\frac{3}{2} + 3\alpha\right) \frac{1}{\#h_{(9)}} \sum_{\chi \pmod{9}} \sum'_{\mathfrak{n} \neq 0} \chi(\mathfrak{n}) \sum_{a|N(\mathfrak{n})} \frac{\mu_{\mathbb{N}}(a)}{a^{\frac{3}{2}+3\alpha}} w\left(\frac{N(\mathfrak{n})}{X}\right).$$

Next notice that the fact that  $\mathfrak{n}$  is squarefree and has no rational divisors means that the divisors of  $N(\mathfrak{n})$  can be parametrized by the norms of the  $\mathbb{Z}[\omega]$  divisors of  $\mathfrak{n}$ . Furthermore, if  $a|\mathfrak{n}$ , then  $\mu_\omega(a) = \mu_{\mathbb{N}}(N(a))$ , so we have

$$(99) \quad M_{\text{res}} = \zeta\left(\frac{3}{2} + 3\alpha\right) \frac{1}{\#h_{(9)}} \sum_{\chi \pmod{9}} \sum'_{\mathfrak{n} \neq 0} \chi(\mathfrak{n}) \sum_{a|\mathfrak{n}} \frac{\mu_\omega(a)}{N(a)^{\frac{3}{2}+3\alpha}} w\left(\frac{N(\mathfrak{n})}{X}\right).$$

Using the Mellin transform of  $w$ , we get

$$(100) \quad \sum'_{\mathfrak{n} \equiv 0 \pmod{a}} \chi(\mathfrak{n}) w\left(\frac{N(n)}{X}\right) = \frac{1}{2\pi i} \int_{(2)} X^s \widetilde{w}(s) \sum'_{\mathfrak{n} \equiv 0 \pmod{a}} \frac{\chi(\mathfrak{n}[a, d])}{N(\mathfrak{n})^s} ds.$$

Now we use Möbius inversion to remove the condition that  $\mathfrak{n}$  has no rational prime divisor to get

$$(101) \quad Z(s) := \sum'_{\mathfrak{n} \equiv 0 \pmod{a}} \frac{\chi(\mathfrak{n})}{N(\mathfrak{n})^s} = \sum_{d \in \mathbb{N}} \mu_{\mathbb{N}}(d) \sum_{\substack{\mathfrak{n} \equiv 0 \pmod{a} \\ \mathfrak{n} \equiv 0 \pmod{d}}} \mu_{\omega}^2(\mathfrak{n}) \frac{\chi(\mathfrak{n})}{N(\mathfrak{n})^s}$$

$$(102) \quad = \sum_{d \in \mathbb{N}} \mu_{\mathbb{N}}(d) \frac{\chi([a, d])}{N([a, d])^s} \sum_{\mathfrak{n} \neq 0} \frac{\chi(\mathfrak{n}) \mu_{\omega}^2(\mathfrak{n}[a, d])}{N(\mathfrak{n})^s}$$

$$(103) \quad = \sum_{d \in \mathbb{N}} \mu_{\mathbb{N}}(d) \frac{\chi([a, d]) \mu_{\omega}^2([a, d])}{N([a, d])^s} \prod_{\mathfrak{p} \nmid [a, d]} (1 + \chi(\mathfrak{p}) N(\mathfrak{p})^{-s})$$

$$(104) \quad = \frac{L(s, \chi)}{L(2s, \chi^2)} \sum_{d \in \mathbb{N}} \mu_{\mathbb{N}}(d) \frac{\chi([a, d]) \mu_{\omega}^2([a, d])}{N([a, d])^s} \prod_{\mathfrak{p} \mid [a, d]} (1 + \chi(\mathfrak{p}) N(\mathfrak{p})^{-s})^{-1}.$$

Note that since both  $a$  and  $d$  are squarefree viewed as elements (or ideals) in  $\mathbb{Z}[\omega]$ , we have  $\mu_{\omega}^2([a, d]) = 1$ .

We then compute

$$(105) \quad \frac{L(2s, \chi^2)}{L(s, \chi)} Z(s) = \sum_{d \in \mathbb{N}} \mu_{\mathbb{N}}(d) \frac{\chi([a, d])}{N([a, d])^s} \prod_{\mathfrak{p} \mid [a, d]} \left(1 + \frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\right)^{-1}$$

$$(106) \quad = \sum_{d \in \mathbb{N}} \mu_{\mathbb{N}}(d) \frac{\chi(ad) \bar{\chi}((a, d)) N((a, d))^s}{N(ad)^s} \prod_{\mathfrak{p} \mid a} \left(1 + \frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\right)^{-1} \prod_{\mathfrak{p} \mid d, \mathfrak{p} \nmid a} \left(1 + \frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\right)^{-1}$$

$$(107) \quad = \frac{\chi(a)}{N(a)^s} \prod_{\mathfrak{p} \mid a} \left(1 + \frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\right)^{-1} \sum_{d \in \mathbb{N}} \mu_{\mathbb{N}}(d) \frac{\chi(d) \bar{\chi}((a, d)) N((a, d))^s}{N(d)^s} \prod_{\mathfrak{p} \mid d, \mathfrak{p} \nmid a} \left(1 + \frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\right)^{-1}$$

$$(108) \quad = \frac{\chi(a)}{N(a)^s} \prod_{\mathfrak{p} \mid a} \left(1 + \frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\right)^{-1} \prod_{(p, a)=1} \left(1 - \frac{\chi(p)}{N(p)^s} \prod_{\mathfrak{p} \mid p} \left(1 + \frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\right)^{-1}\right) \\ \prod_{(p, a) \neq 1} \left(1 - \frac{\chi(p) \bar{\chi}(a, p) N((a, p))^s}{N(p)^s} \prod_{\mathfrak{p} \mid p, \mathfrak{p} \nmid a} \left(1 + \frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\right)^{-1}\right)$$

It should now be clear that the above Dirichlet series has analytic continuation to  $\operatorname{Re}(s) > 1/2 + \varepsilon$ , and is bounded by  $N(a)^\varepsilon$  in this region. We therefore obtain that  $Z(s)$  has a pole at  $s = 1$  when  $\chi$  is the principal character, and is bounded by  $N(a)^\varepsilon$  in this region. Thus we get that  $M_{\text{res}}$  itself equals the residue at  $s = 1$  plus an error term of size  $O(X^{1/2+\varepsilon})$ . It is also not hard to see that the residue is a positive constant that can be expressed as an Euler product, if desired.

In summary, we have shown

**Lemma 6.1.** *There exists a constant  $c > 0$  such that*

$$(109) \quad M_0 = cX + O(X^{1+\varepsilon} A^{-1/2} + X^{1/2+\varepsilon}).$$

Remark. In Section 7 we take  $A = X^{4/7} N(r)^{4/7} = X^{4/7}$  (since we take  $r = 1$  when evaluating the main term), which leads to an error here of size  $\ll X^{5/7+\varepsilon}$ .

## 7. OPTIMIZATION AND RECURSION

The authors recommend the use of a computer to perform the calculations in this section. Recall that under the assumption (41),

$$(110) \quad M'_1 \ll X^\varepsilon \left( X^{\frac{\gamma}{2\gamma-\delta}} A^{\frac{3(2\gamma-\delta-1)}{2(2\gamma-\delta)}} N(r)^{\frac{\delta}{2\gamma-\delta}} N(a)^{\frac{\nu-\frac{1}{2}}{2\gamma-\delta}+\frac{1}{2}} + X^{1/2} A^{3/4} N(a)^{1/4} \right),$$

and (independently of (41)),

$$(111) \quad M_2 \ll X^\varepsilon \left( X^{1/2} B N(a)^{1/2} + B^{1/6} X^{5/6} \right).$$

We choose  $A := X^{4/7} N(r)^{4/7}$  and thus  $B = X^{3/7} N(r)^{3/7}$ . Now from (110) and (111), we obtain

$$(112) \quad M'_1 + M_2 \ll X^\varepsilon \left( X^{\frac{19\gamma-6\delta-6}{7(2\gamma-\delta)}} N(r)^{\frac{12\gamma+\delta-6}{7(2\gamma-\delta)}} N(a)^{\frac{\nu-\frac{1}{2}}{2\gamma-\delta}+\frac{1}{2}} + X^{13/14} N(r)^{3/7} N(a)^{1/2} \right),$$

noting that the term of size  $X^{19/21} N(r)^{1/14}$  coming from  $B^{1/6} X^{5/6}$  is dominated by the second term on the right-hand side of (112). As an initial estimate, we use

$$(113) \quad M'_1 + M_2 \ll X^\varepsilon \left( X^{10/9} + X^{13/12} N(r)^{1/4} \right)$$

which follows from (31). Using the *two* triples  $(\gamma, \delta, \nu) = (10/9, 0, 0)$  and  $(\gamma, \delta, \nu) = (13/12, 1/4, 0)$  in (41), we obtain

$$(114) \quad M'_1 + M_2 \ll X^\varepsilon \left( X^{\frac{157}{161}} N(r)^{\frac{87}{161}} N(a)^{\frac{11}{46}} + X^{\frac{34}{35}} N(r)^{\frac{33}{70}} N(a)^{\frac{11}{40}} + X^{\frac{13}{14}} N(r)^{\frac{3}{7}} N(a)^{\frac{1}{2}} \right).$$

We see that the right-hand side is bounded by

$$(115) \quad \ll X^{157/161} N(r)^{87/161} N(a)^{1/2}$$

and thus get the new triple  $(\gamma, \delta, \nu) = (157/161, 87/161, 1/2)$ . After one more recursion, we obtain the triple  $(1495/1589, 1005/1589, 1/2)$ , and one more recursion leads to the triple  $(13/14, 9411/13895, 1/2)$ , where  $\gamma = 13/14$  now arises from the second term in (112) (in the first term we have the power  $\gamma = 12841/13895 < 13/14$ ). Plugging in  $a = r = 1$  gives the bound  $X^{13/14+\varepsilon}$ , and gives the error term in Theorem 1.1. This completes the proof of Theorem 1.1 except for the proof of Theorem 3.1 which we provide in Section 14.

## 8. THE CUBIC LARGE SIEVE

Next we shall establish our cubic large sieve, Theorem 1.4.

It is easy to reduce the expression on the left-hand side of (4) to a sum of similar expressions with the additional summation conditions  $(q, 3) = 1$  and  $(m, 3) = 1$  included. Thus it suffices to estimate

$$(116) \quad \sum_{\substack{Q < q \leq 2Q \\ (q, 3) = 1}} \sum_{\substack{\chi \pmod{q} \\ \chi^3 = \chi_0}}^* \left| \sum_{\substack{M < m \leq 2M \\ (m, 3) = 1}}^* a_m \chi(m) \right|^2 = \sum'_{\substack{n \in \mathbb{Z}[\omega] \\ Q < N(n) \leq 2Q \\ n \equiv 1 \pmod{3}}} \left| \sum_{\substack{M < m \leq 2M \\ (m, 3) = 1}}^* a_m \chi_n(m) \right|^2 \\ = \frac{1}{2} \sum'_{\substack{n \in \mathbb{Z}[\omega] \\ Q < N(n) \leq 2Q \\ n \equiv \pm 1 \pmod{3}}} \left| \sum_{\substack{M < m \leq 2M \\ (m, 3) = 1}}^* a_m \chi_n(m) \right|^2,$$

where the prime indicates that  $n$  is square-free and has no rational prime divisor. The reader should keep in mind that

$$(117) \quad \chi_n(m) = \left(\frac{m}{n}\right)_3,$$

the cubic residue symbol. We shall use this notation for *all*  $n \in \mathbb{Z}[\omega]$  and  $m \in \mathbb{Z}$ .

In the following, we shall estimate the expression in the last line of (116). To this end, we will frequently make use of ideas and results in [Hea1] and [Hea2], where (7) and (8) were established, respectively (in particular, we shall use (8) itself). However, here we have to manage the additional difficulty lying in the asymmetry of the sums over  $m$  and  $n$ . This will require some new ideas. In particular, we shall use Hölder's inequality to enlarge the sum over  $m$  and *two* versions of the Poisson summation formula: the one-dimensional version for the sum over  $m \in \mathbb{Z}$  and the two-dimensional version for the sum over  $n \in \mathbb{Z}[\omega]$ .

We begin by defining a norm corresponding to the double sum in the last line of (116) by

$$(118) \quad B_1(Q, M) := \sup_{(a_m)} ||a_m||^{-2} \sum'_{\substack{n \in \mathbb{Z}[\omega] \\ Q < N(n) \leq 2Q \\ n \equiv \pm 1 \pmod{3}}} \left| \sum_{\substack{M < m \leq 2M \\ (m,3)=1}}^* a_m \chi_n(m) \right|^2,$$

where

$$(119) \quad ||a_m||^2 = \sum_m |a_m|^2,$$

and where by convention we suppose that  $(a_m)$  is not identically zero.

We further introduce the norm

$$(120) \quad B_2(Q, M) := \sup_{(a_m)} ||a_m||^{-2} \sum_{\substack{n \in \mathbb{Z}[\omega] \\ Q < N(n) \leq 2Q \\ n \equiv \pm 1 \pmod{3}}}^* \left| \sum_{\substack{M < m \leq 2M \\ (m,3)=1}}^* a_m \chi_n(m) \right|^2,$$

where the star attached to the sum over  $n$  indicates that  $n$  is square-free, and the norm

$$(121) \quad B_3(Q, M) := \sup_{(a_m)} ||a_m||^{-2} \sum_{\substack{n \in \mathbb{Z}[\omega] \\ Q < N(n) \leq 2Q}} \left| \sum_{\substack{M < m \leq 2M \\ (m,3)=1}}^* a_m \chi_n(m) \right|^2.$$

As in [Hea2], we introduce an infinitely differentiable weight function  $W : \mathbb{R} \rightarrow \mathbb{R}$ , defined by

$$(122) \quad W(x) = \begin{cases} \exp\left(-\frac{1}{(2x-1)(5-2x)}\right), & \text{if } \frac{1}{2} < x < \frac{5}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

It follows that the double sum on the right-hand side of (121) is bounded by

$$(123) \quad \sum_{\substack{n \in \mathbb{Z}[\omega] \\ Q < N(n) \leq 2Q}} \left| \sum_{\substack{M < m \leq 2M \\ (m,3)=1}}^* a_m \chi_n(m) \right|^2 \ll \sum_{n \in \mathbb{Z}[\omega]} W\left(\frac{N(n)}{Q}\right) \left| \sum_{\substack{M < m \leq 2M \\ (m,3)=1}}^* a_m \chi_n(m) \right|^2.$$

Expanding the sum on the right-hand side, we obtain

$$(124) \quad \sum_{\substack{M < m_1 \leq 2M \\ M < m_2 \leq 2M \\ (m_1 m_2, 3) = 1}}^* a_{m_1} \overline{a_{m_2}} \sum_{n \in \mathbb{Z}[\omega]} W\left(\frac{N(n)}{Q}\right) \chi_n(m_1) \overline{\chi_n}(m_2).$$

As in [Hea2], it turns out that we may restrict attention to the case in which  $m_1$  and  $m_2$  are coprime. We define another norm  $B_4$  corresponding to the above sum with the restriction  $(m_1, m_2) = 1$  included by

$$(125) \quad B_4(Q, M) := \sup_{(a_m)} \|a_m\|^{-2} \sum_{\substack{M < m_1, m_2 \leq 2M \\ (m_1 m_2, 3) = 1 \\ (m_1, m_2) = 1}}^* a_{m_1} \overline{a_{m_2}} \sum_{n \in \mathbb{Z}[\omega]} W\left(\frac{N(n)}{Q}\right) \chi_n(m_1) \overline{\chi_n}(m_2).$$

In the following, we compare the norms  $B_i(Q, M)$ ,  $i = 1, \dots, 4$ .

## 9. COMPARISON OF THE NORMS

The norm  $B_1(Q, M)$  satisfies the useful property that it is essentially increasing in  $Q$ . More precisely, there exists an absolute constant  $C \geq 1$  as follows. Let  $Q_1, M \gg 1$  and  $Q_2 \geq CQ_1 \log(2Q_1 M)$ , then

$$(126) \quad B_1(Q_1, M) \ll B_1(Q_2, M).$$

Although this can be proved in essentially the same way as Lemma 9 in [Hea1], we include some details of the proof to keep our paper as self-contained as possible.

Write  $K = Q_2/Q_1$ . Let  $\pi$  be a non-rational prime in  $\mathbb{Z}[\omega]$  satisfying  $\pi \equiv \pm 1 \pmod{3}$  and  $\frac{2}{3}K < N(\pi) < \frac{4}{3}K$ . Then, similarly as in [Hea1], we observe that

$$(127) \quad \sum_n' \left| \sum_m^* a_m \chi_n(m) \right|^2 \leq \sum_{\pi|n}' \left| \sum_m^* a_m \chi_n(m) \right|^2 + 2 \sum_{\pi \nmid n}' \left| \sum_{\pi \nmid m}^* a_m \chi_n(m) \right|^2 + 2 \sum_{\pi \nmid n}' \left| \sum_{\pi|m}^* a_m \chi_n(m) \right|^2,$$

by Cauchy-Schwarz. Estimating the second double sum on the right-hand side by writing

$$(128) \quad \sum_{\pi \nmid m}^* a_m \chi_n(m) = \sum_{\pi \nmid m}^* b_m \chi_{\pi n}(m) \quad \text{with } b_m = \overline{\chi_\pi}(m) a_m$$

and using the definition of  $B_1(Q_2, N)$ , and the third double sum by using the definition of  $B_1(Q_1, N)$ , we obtain, similarly as in [Hea1],

$$(129) \quad \sum_{\substack{n \in \mathbb{Z}[\omega] \\ \alpha Q_1 < N(n) \leq \beta Q_1 \\ n \equiv \pm 1 \pmod{3}}} \left| \sum_{\substack{M < m \leq 2M \\ (m, 3) = 1}}^* a_m \chi_n(m) \right|^2 \ll \sum_{\substack{n \in \mathbb{Z}[\omega] \\ \alpha Q_1 < N(n) \leq \beta Q_1 \\ n \equiv \pm 1 \pmod{3}}} \left| \sum_{\substack{M < m \leq 2M \\ (m, 3) = 1}}^* a_m \chi_n(m) \right|^2 + B_1(Q_2, N) \sum_m^* |a_m|^2 + B_1(Q_1, N) \sum_{\pi|m}^* |a_m|^2,$$

where  $(\alpha, \beta)$  is either  $(1, 3/2)$  or  $(3/2, 2)$ , and  $N(\pi)$  lies in the range  $(K, \frac{4}{3}K)$  or  $(\frac{2}{3}K, K)$ , respectively. We now sum over all relevant primes  $\pi$ . We note that the number of available primes is  $\asymp K/\log K$ , and that the number of prime factors  $\pi \in \mathbb{Z}[\omega]$  with  $\frac{2}{3}K < N(\pi) < \frac{4}{3}K$  of every of the relevant  $m$ 's and  $n$ 's is bounded by  $O(\log(2Q_1N)/\log K)$ . Now a short calculation like in [Hea1] gives

$$(130) \quad \frac{K}{\log K} \sum'_{\substack{n \in \mathbb{Z}[\omega] \\ Q_1 < N(n) \leq 2Q_1 \\ n \equiv \pm 1 \pmod{3}}} \left| \sum_{\substack{M < m \leq 2M \\ (m,3)=1}}^* a_m \chi_n(m) \right|^2 \\ \ll \left( \frac{\log(2Q_1N)}{\log K} B_1(Q_1, N) + \frac{K}{\log K} B_1(Q_2, N) \right) \|a_m\|^2.$$

We choose the coefficients  $a_m$  so that

$$(131) \quad \sum'_{\substack{n \in \mathbb{Z}[\omega] \\ Q_1 < N(n) \leq 2Q_1 \\ n \equiv \pm 1 \pmod{3}}} \left| \sum_{\substack{M < m \leq 2M \\ (m,3)=1}}^* a_m \chi_n(m) \right|^2 = B_1(Q_1, N) \|a_m\|^2.$$

Thus if  $K \geq C \log(2Q_1N)$  with  $C$  a sufficiently large absolute constant, we deduce that

$$(132) \quad B_1(Q_1, N) \ll B_1(Q_2, N),$$

as claimed.

Next, we compare  $B_1$  and  $B_2$ . Trivially,  $B_1(Q, M) \leq B_2(Q, M)$ . Conversely, we prove an estimate of  $B_2$  in terms of  $B_1$ . We have

$$(133) \quad \sum_{\substack{n \in \mathbb{Z}[\omega] \\ Q < N(n) \leq 2Q \\ n \equiv \pm 1 \pmod{3}}}^* \left| \sum_{\substack{M < m \leq 2M \\ (m,3)=1}}^* a_m \chi_n(m) \right|^2 = \sum_{\substack{k \in \mathbb{N} \\ N(k) \leq 2Q \\ (k,3)=1}}^* \sum'_{\substack{n \in \mathbb{Z}[\omega] \\ Q/N(k) < N(n) \leq 2Q/N(k) \\ (k,n)=1 \\ n \equiv \pm 1 \pmod{3}}} \left| \sum_{\substack{M < m \leq 2M \\ (m,3k)=1}}^* a_m \chi_n(m) \right|^2$$

$$(134) \quad \leq \sum_{1 \leq k \leq \sqrt{2Q}}^* \sum'_{\substack{n \in \mathbb{Z}[\omega] \\ Q/k^2 < N(n) \leq 2Q/k^2 \\ n \equiv \pm 1 \pmod{3}}} \left| \sum_{\substack{M < m \leq 2M \\ (m,3k)=1}}^* a_m \chi_n(m) \right|^2.$$

Breaking the outer sum in the last line into  $O(\log 2Q)$  dyadic intervals, we find that the expression in the last line is

$$(135) \ll \log(2Q) \sup_{1 \leq X \leq Q} \sum_{Q^{1/2}X^{-1/2} \leq k \leq 2Q^{1/2}X^{-1/2}}^* \sum_{\substack{n \in \mathbb{Z}[\omega] \\ X/4 < N(n) \leq 2X \\ n \equiv \pm 1 \pmod{3}}}^' \left| \sum_{\substack{M < m \leq 2M \\ (m,3)=1}}^* a_m \chi_n(m) \right|^2$$

$$(136) \ll (\log 2Q) \sup_{1 \leq X \leq Q} Q^{1/2}X^{-1/2} (B_1(X/4, M) + B_1(X/2, M) + B_1(X, M)) \|a_m\|^2$$

$$(137) \ll (\log 2Q) \sup_{1/2 \leq X \leq Q} Q^{1/2}X^{-1/2} B_1(X, M) \|a_m\|^2.$$

Hence, we have

$$(138) \quad B_2(Q, M) \ll (\log 2Q) Q^{1/2} X^{-1/2} B_1(X, M)$$

for a suitable  $X$  with  $1 \ll X \ll Q$ .

Next, we compare  $B_2$  and  $B_3$ . Trivially,  $B_2(Q, M) \leq B_3(Q, M)$ . On the other hand, by a similar argument as used in the proof of Lemma 6 in [Hea2], one can prove that

$$(139) \quad B_3(Q, M) \ll (\log Q)^2 Q^{1/3} X^{-1/3} Y^{-2/3} \min\{Y B_2(X, M), X B_2(Y, M)\}$$

for suitable  $X, Y \gg 1$  with  $XY^2 \ll Q$ . For our purposes, a simpler estimate suffices which we derive in the following.

We write  $n$  in the form  $n = ab^2c$ , where  $a$  is square-free,  $a \equiv \pm 1 \pmod{3}$  and  $c$  is a product of a power of  $\sqrt{-3}$  and a unit  $u$ . Then it follows that

$$(140) \quad \sum_{\substack{n \in \mathbb{Z}[\omega] \\ Q < N(n) \leq 2Q}} \left| \sum_{\substack{M < m \leq 2M \\ (m,3)=1}}^* a_m \chi_n(m) \right|^2 \\ \leq \sum_{\substack{c \in \mathbb{Z}[\omega] \\ c = (\sqrt{-3})^k u}} \sum_{\substack{b \in \mathbb{Z}[\omega] \\ N(b) \leq \sqrt{2Q/3^k}}} \sum_{\substack{a \in \mathbb{Z}[\omega] \\ Q/(3^k N(b)^2) < a \leq 2Q/(3^k N(b)^2) \\ a \equiv \pm 1 \pmod{3}}}^* \left| \sum_{\substack{M < m \leq 2M \\ (m,3)=1}}^* a_m \chi_{b^2c}(m) \chi_a(m) \right|^2.$$

Breaking the second sum on the right-hand side into dyadic intervals, we find that the expression on the right-hand side is

$$\begin{aligned} &\ll (\log 2Q) \sum_{\substack{c \in \mathbb{Z}[\omega] \\ c = (\sqrt{-3})^k u}} \sup_{1 \leq X \leq Q/3^k} \sum_{\substack{b \in \mathbb{Z}[\omega] \\ \sqrt{Q/3^k X} \leq N(b) \leq 2\sqrt{Q/3^k X}}} \sum_{\substack{a \in \mathbb{Z}[\omega] \\ X/4 < N(a) \leq 2X \\ a \equiv \pm 1 \pmod{3}}}^* \left| \sum_{\substack{M < m \leq 2M \\ (m,3)=1}}^* a_m \chi_{b^2c}(m) \chi_a(m) \right|^2 \\ &\leq (\log 2Q) \sum_{\substack{c \in \mathbb{Z}[\omega] \\ c = (\sqrt{-3})^k u}} \sup_{1 \leq X \leq Q/3^k} Q^{\frac{1}{2}} X^{-\frac{1}{2}} \cdot (B_2(\frac{X}{4}, M) + B_2(\frac{X}{2}, M) + B_2(X, M)) \|a_m\|^2 \\ &\ll (\log 2Q)^2 \sup_{1/2 \leq X \leq Q} Q^{1/2} X^{-1/2} B_2(X, M) \|a_m\|^2. \end{aligned}$$

Hence, we have

$$(141) \quad B_3(Q, M) \ll (\log 2Q)^2 Q^{1/2} X^{-1/2} B_2(X, M)$$

for a suitable  $X$  with  $1 \ll X \ll Q$ . This estimate can also be deduced directly from (139) using only the first term in the minimum on the right-hand side of (139) and the bound  $XY^2 \ll Q$ .

Combining (138) and (141), we obtain a bound for  $B_3$  in terms of  $B_1$ , namely

$$(142) \quad B_3(Q, M) \ll (\log 2Q)^3 Q^{1/2} X^{-1/2} B_1(X, M)$$

for a suitable  $X$  with  $1 \ll X \ll Q$ . Conversely, we have the trivial bound

$$(143) \quad B_1(Q, M) \ll B_3(Q, M).$$

Finally, we compare  $B_3$  and  $B_4$ . First, we show that there exist positive integers  $\Delta_1, \Delta_2$  with  $\Delta_2^2 \geq \Delta_1$  such that

$$(144) \quad B_3(Q, M) \ll M^\varepsilon B_4\left(\frac{Q}{\Delta_1}, \frac{M}{\Delta_2}\right).$$

Since the proof is essentially the same as that of Lemma 7 in [Hea2], we will only give a sketch.

As noted in the previous section, the expression in (124) dominates the double sum on the right-hand side of (121) corresponding to the norm  $B_3$ . Sorting the terms in (124) according to  $\delta = (m_1, m_2)$ , and detecting the condition  $(n, \delta) = 1$  by the Möbius function for  $\mathbb{Z}[\omega]$ , we obtain that the expression in (124) equals

$$(145) \quad \sum_{(\delta, 3)=1}^* \sum_{d|\delta} \mu_\omega(d) \sum_{s \in \mathbb{Z}[\omega]} W\left(\frac{N(s)}{Q/N(d)}\right) \sum_{\substack{M/\delta < r_1 \leq 2M/\delta \\ M/\delta < r_2 \leq 2M/\delta \\ (r_1 r_2, 3\delta)=1 \\ (r_1, r_2)=1}}^* a_{r_1}^* \overline{a_{r_2}^*} \chi_s(r_1) \overline{\chi_s(r_2)},$$

where  $d$  runs over non-associate divisors of  $\delta$  in  $\mathbb{Z}[\omega]$ , and

$$(146) \quad a_r^* := a_{r\delta} \chi_d(r).$$

By the definition of  $B_4$ , the expression in (145) is bounded by

$$(147) \quad \sum_{(\delta, 3)=1}^* \sum_{d|\delta} B_4\left(\frac{Q}{N(d)}, \frac{M}{\delta}\right) \sum_{\substack{M/\delta < r \leq 2M/\delta \\ (r, 3\delta)=1}}^* |a_{r\delta}|^2.$$

From this, it is easy to deduce the desired estimate (144) in a similar way as in [Hea2] upon noting that  $\delta^2 = N(\delta) \geq N(d)$ .

To derive a bound of  $B_4$  in terms of  $B_3$ , we remember that  $\chi_n(m) = \left(\frac{m}{n}\right)_3$  and employ Lemma 10 in [Hea2] with the residue symbol reversed. This lemma is based on the two-dimensional Poisson summation formula for  $\mathbb{Z}[\omega]$ . We note that

$$(148) \quad \tilde{\chi}(n) = \left(\frac{m_1}{n}\right)_3 \overline{\left(\frac{m_2}{n}\right)_3}$$

is a primitive character to modulus  $m_1 m_2$  provided that  $m_1, m_2$  are coprime to each other and to 3, and are square-free. We further note that  $N(m_1 m_2) = (m_1 m_2)^2$  since  $m_1, m_2 \in \mathbb{Z}$ .



For the sum corresponding to the norm  $B_4$ , we thus obtain

$$\begin{aligned}
 (149) \quad & \sum_{\substack{M < m_1, m_2 \leq 2M \\ (m_1 m_2, 3) = 1 \\ (m_1, m_2) = 1}}^* a_{m_1} \overline{a_{m_2}} \sum_{n \in \mathbb{Z}[\omega]} W \left( \frac{N(n)}{Q} \right) \left( \frac{m_1}{n} \right)_3 \overline{\left( \frac{m_2}{n} \right)_3} \\
 &= Q \sum_{k \in \mathbb{Z}[\omega]} \sum_{\substack{M < m_1, m_2 \leq 2M \\ (m_1 m_2, 3) = 1 \\ (m_1, m_2) = 1}}^* b_{m_1} \overline{b_{m_2}} \widetilde{W} \left( \frac{N(k)Q}{(m_1 m_2)^2} \right) \overline{\left( \frac{m_1}{k} \right)_3} \left( \frac{m_2}{k} \right)_3,
 \end{aligned}$$

with  $\widetilde{W}$  being a certain weight function of rapid decay (related to the Fourier transform of  $W$ ), and

$$(150) \quad b_m := a_m \left( \frac{m}{\sqrt{-3}} \right)_3 \frac{g(m)}{m^2},$$

where recall  $g(m)$  is a Gauss sum described in Section 2.3. Here  $d$  runs over all residue classes modulo  $m$  in  $\mathbb{Z}[\omega]$ .

Now, similarly as in [Hea2], we remove the weight function  $\widetilde{W}$  by using the Mellin transform and pick out the coprimality condition  $(m_1, m_2) = 1$  by using the Möbius function to separate the variables  $m_1, m_2$ . We then use the Cauchy-Schwarz inequality, and after a short calculation like in [Hea2], we arrive at the estimate

$$\begin{aligned}
 (151) \quad B_4(Q, M) &\ll Q M^{8\varepsilon-2} \max \{ B_3(K, M) : K \leq M^4 Q^{-1} \} + \\
 &Q^{-1} M^{6+8\varepsilon} \sum_{K > M^4/Q} K^{-2-\varepsilon} B_3(K, M) \quad \text{if } M \geq 1,
 \end{aligned}$$

where  $K$  runs over powers of 2. This corresponds to Lemma 8 in [Hea2]. We also have the trivial bound

$$(152) \quad B_4(Q, M) \ll Q \quad \text{if } M < 1.$$

Combining this with (151), we get

$$\begin{aligned}
 (153) \quad B_4(Q, M) &\ll Q + Q M^{8\varepsilon-2} \max \{ B_3(K, M) : K \leq M^4 Q^{-1} \} + \\
 &Q^{-1} M^{6+8\varepsilon} \sum_{K > M^4/Q} K^{-2-\varepsilon} B_3(K, M)
 \end{aligned}$$

for any  $Q, M > 0$ .

## 10. DEFINITION OF DUAL NORMS

We now define two dual norms which we then compare and estimate. First, we define a norm  $C_1(M, Q)$  by

$$(154) \quad C_1(M, Q) := \sup_{(b_n)} \|b_n\|^{-2} \sum_{\substack{M < m \leq 2M \\ (m, 3) = 1}}^* \left| \sum_{\substack{n \in \mathbb{Z}[\omega] \\ Q < N(n) \leq 2Q \\ n \equiv \pm 1 \pmod{3}}} b_n \chi_n(m) \right|^2.$$

By the duality principle, we have

$$(155) \quad B_1(Q, M) = C_1(M, Q).$$

Furthermore, we define the norm

$$(156) \quad C_2(M, Q) := \sup_{(b_n)} \|b_n\|^{-2} \sum_{M < m \leq 2M} \left| \sum'_{\substack{n \in \mathbb{Z}[\omega] \\ Q < N(n) \leq 2Q \\ n \equiv \pm 1 \pmod{3}}} b_n \chi_n(m) \right|^2.$$

Trivially, we have

$$(157) \quad C_1(M, Q) \leq C_2(M, Q).$$

## 11. A BOUND FOR $C_2$ IN TERMS OF $C_2$

In the following, we prove the estimate

$$(158) \quad C_2(M, Q) \ll M^\varepsilon Q^{1-1/v} \sum_{j=0}^{v-1} C_2(M, 2^j Q^v)^{1/v}$$

for any  $v \in \mathbb{N}$ . We define a dual norm by

$$(159) \quad C'_2(Q, M) := \sup_{(a_m)} \|a_m\|^{-2} \sum'_{\substack{n \in \mathbb{Z}[\omega] \\ Q < N(n) \leq 2Q \\ n \equiv \pm 1 \pmod{3}}} \left| \sum_{M < m \leq 2M} a_m \chi_n(m) \right|^2.$$

By the duality principle, we have

$$(160) \quad C'_2(Q, M) = C_2(M, Q).$$

Assume  $(c_m)$  is a sequence such that

$$(161) \quad C'_2(Q, M) = \|c_m\|^{-2} \sum'_{\substack{n \in \mathbb{Z}[\omega] \\ Q < N(n) \leq 2Q \\ n \equiv \pm 1 \pmod{3}}} \left| \sum_{M < m \leq 2M} c_m \chi_n(m) \right|^2.$$

By Hölder's inequality and multiplicativity of the residue symbol, we have

$$(162) \quad \begin{aligned} C'_2(Q, M) &\ll \|c_m\|^{-2} Q^{1-1/v} \left( \sum'_{\substack{n \in \mathbb{Z}[\omega] \\ Q < N(n) \leq 2Q \\ n \equiv \pm 1 \pmod{3}}} \left| \sum_{M < m \leq 2M} c_m \chi_n(m) \right|^{2v} \right)^{1/v} \\ &= \|c_m\|^{-2} Q^{1-1/v} \left( \sum'_{\substack{n \in \mathbb{Z}[\omega] \\ Q < N(n) \leq 2Q \\ n \equiv \pm 1 \pmod{3}}} \left| \sum_{M^v < m \leq (2M)^v} d_m \chi_n(m) \right|^2 \right)^{1/v}, \end{aligned}$$

where

$$(163) \quad d_m = \sum_{\substack{M < m_1, \dots, m_v \leq 2M \\ m_1 \cdots m_v = m}} c_{m_1} \cdots c_{m_v}.$$

By splitting the sum over  $m$  into dyadic segments, we have

$$(164) \quad C'_2(Q, M) \ll Q^{1-1/v} \cdot \sum_{j=0}^{v-1} \|c_m\|^{-2} \left( \sum_{2^j M^v < m \leq 2^{j+1} M^v} |d_m|^2 \right)^{1/v} C'_2(Q, 2^j M^v)^{1/v}.$$

Using the Cauchy-Schwarz inequality and the bound

$$(165) \quad \sum_{\substack{m_1, \dots, m_v \in \mathbb{N} \\ m_1 \cdots m_v = n}} 1 \ll m^\varepsilon,$$

we obtain

$$(166) \quad \sum_{m \in \mathbb{N}} |d_m|^2 \ll M^\varepsilon \sum_{m \in \mathbb{N}} \sum_{\substack{M < m_1, \dots, m_v \leq 2M \\ m_1 \cdots m_v = m}} |c_{m_1} \cdots c_{m_v}|^2$$

$$(167) \quad = M^\varepsilon \left( \sum_{M < m \leq 2M} |c_m|^2 \right)^v.$$

Hence,

$$(168) \quad C'_2(Q, M) \ll M^\varepsilon Q^{1-1/v} \sum_{j=0}^{v-1} C'_2(Q, 2^j M^v)^{1/v} = M^\varepsilon Q^{1-1/v} \sum_{j=0}^{v-1} C_2(2^j M^v, Q)^{1/v},$$

which proves our claim.

## 12. ESTIMATING $C_2$

Recall  $C_2(M, Q)$  is the norm of the sum

$$(169) \quad \sum_{M < m \leq 2M} \left| \sum'_{\substack{n \in \mathbb{Z}[\omega] \\ Q < N(n) \leq 2Q \\ n \equiv \pm 1 \pmod{3}}} b_n \chi_n(m) \right|^2,$$

where the prime indicates that  $m$  is square-free and has no rational prime divisor.

We will estimate  $C_2(M, Q)$  directly using Theorem 2 in [Hea2]. Our result will be

$$(170) \quad C_2(M, Q) \ll (QM)^\varepsilon (M + Q^{5/3}).$$

For comparison, the ordinary large sieve inequality gives the bound

$$(171) \quad C_2(M, Q) \ll M + Q^2.$$

By using (155), (157) and (158), we then deduce

$$(172) \quad B_1(Q, M) \ll (QM)^\varepsilon (Q^{1-1/v} M + Q^{1+2/(3v)})$$

for any  $v \in \mathbb{N}$ .

The sum in (169) is obviously bounded by

$$(173) \quad \ll \sum_{m \in \mathbb{Z}} W\left(\frac{m}{M}\right) \left| \sum'_n b_n \chi_n(m) \right|^2,$$

where the weight function  $W$  is defined as in (122). Expanding out the sum in (173) we get

$$(174) \quad \sum'_{\substack{n_1, n_2 \in \mathbb{Z}[\omega] \\ Q < N(n_1), N(n_2) \leq 2Q \\ n_1, n_2 \equiv \pm 1 \pmod{3}}} b_{n_1} \overline{b_{n_2}} \sum_{m \in \mathbb{Z}} W\left(\frac{m}{M}\right) \chi_{n_1} \overline{\chi_{n_2}}(m).$$

Now we extract the greatest common divisor  $\Delta$  of  $n_1$  and  $n_2$ , getting

$$(175) \quad \sum'_{\substack{N(\Delta) \leq 2Q \\ \Delta \equiv \pm 1 \pmod{3}}} \sum'_{\substack{n_1, n_2 \in \mathbb{Z}[\omega] \\ \frac{Q}{N(\Delta)} < N(n_1), N(n_2) \leq \frac{2Q}{N(\Delta)} \\ n_1, n_2 \equiv \pm 1 \pmod{3} \\ (n_1, n_2) = 1 \\ (n_1 n_2, \Delta \overline{\Delta}) = 1}} b_{n_1 \Delta} \overline{b_{n_2 \Delta}} \sum_{\substack{m \in \mathbb{Z} \\ (m, \Delta) = 1}} W\left(\frac{m}{M}\right) \chi_{n_1} \overline{\chi_{n_2}}(m).$$

Now we write  $\delta = (n_1, \overline{n_2})$  and change variables via  $n_1 \rightarrow \delta n_1$ ,  $n_2 \rightarrow \overline{\delta} n_2$  to get

$$(176) \quad \sum'_{\substack{N(\Delta) \leq 2Q \\ \Delta \equiv \pm 1 \pmod{3}}} \sum'_{\substack{N(\delta) \leq \frac{2Q}{N(\Delta)} \\ \delta \equiv \pm 1 \pmod{3} \\ (N(\delta), N(\Delta)) = 1}} \sum'_{\substack{n_1, n_2 \in \mathbb{Z}[\omega] \\ \frac{Q}{N(\delta \Delta)} < N(n_1), N(n_2) \leq \frac{2Q}{N(\delta \Delta)} \\ n_1, n_2 \equiv \pm 1 \pmod{3} \\ (N(n_1), N(n_2)) = 1 \\ (n_1 n_2, \delta \overline{\delta} \Delta \overline{\Delta}) = 1}} b_{n_1 \Delta \delta} \overline{b_{n_2 \Delta \overline{\delta}}} \times \\ \sum_{\substack{m \in \mathbb{Z} \\ (m, \Delta) = 1}} W\left(\frac{m}{M}\right) \chi_{n_1} \overline{\chi_{n_2 \delta}}(m),$$

where we use that  $\chi_\delta \overline{\chi_{\delta}} = \chi_\delta^2 = \overline{\chi_\delta}$ . Next we remove the coprimality condition in the sum over  $m$  by the Möbius function, getting

$$(177) \quad \sum_{\substack{m \in \mathbb{Z} \\ (m, \Delta) = 1}} W\left(\frac{m}{M}\right) \chi_{n_1} \overline{\chi_{n_2 \delta}}(m) = \sum_{l | \Delta} \mu(l) \chi_{n_1} \overline{\chi_{n_2 \delta}}(l) \sum_{m \in \mathbb{Z}} W\left(\frac{m}{M/l}\right) \chi_{n_1} \overline{\chi_{n_2 \delta}}(m),$$

which by the Poisson summation formula is

$$(178) \quad \sum_{l | \Delta} \mu(l) \chi_{n_1} \overline{\chi_{n_2 \delta}}(l) \frac{M}{l N(n_1 n_2 \delta)} \sum_{h \in \mathbb{Z}} \widehat{W}\left(\frac{h M}{l N(n_1 n_2 \delta)}\right) \\ \sum_{r \pmod{N(n_1 n_2 \delta)}} \chi_{n_1} \overline{\chi_{n_2 \delta}}(r) e\left(\frac{r h}{N(n_1 n_2 \delta)}\right).$$

When  $h = 0$ , the expression in (178) vanishes unless  $n_1 = n_2 = \delta = 1$ . Hence, the contribution of  $h = 0$  to (176) is

$$(179) \quad \ll M^{1+\varepsilon} \sum'_{\substack{Q < N(\Delta) \leq 2Q \\ \Delta \equiv \pm 1 \pmod{3}}} |b_\Delta|^2 \ll M^{1+\varepsilon} ||b||^2.$$

In the following, we assume that  $h \neq 0$ . The sum over  $r$  in (178) is a Gauss sum that can be computed by writing  $r = r_1 N(n_2 \delta) + r_2 N(n_1 \delta) + r_3 N(n_1 n_2)$  to get

$$\begin{aligned}
 (180) \quad & \sum_{r \pmod{N(n_1 n_2 \delta)}} \chi_{n_1} \overline{\chi_{n_2 \delta}}(r) e\left(\frac{rh}{N(n_1 n_2 \delta)}\right) \\
 &= \sum_{r_1 \pmod{N(n_1)}} \chi_{n_1}(r_1 N(n_2 \delta)) e\left(\frac{r_1 h}{N(n_1)}\right) \sum_{r_2 \pmod{N(n_2)}} \overline{\chi_{n_2}}(r_2 N(n_1 \delta)) e\left(\frac{r_2 h}{N(n_2)}\right) \times \\
 & \quad \sum_{r_3 \pmod{N(\delta)}} \overline{\chi_{\delta}}(r_3 N(n_1 n_2)) e\left(\frac{r_3 h}{N(\delta)}\right) \\
 &= \overline{\chi_{n_1}}(h) \chi_{n_2 \delta}(h) \chi_{n_1}(N(n_2 \delta)) \overline{\chi_{n_2}}(N(n_1 \delta)) \overline{\chi_{\delta}}(N(n_1 n_2)) \tau(\chi_{n_1}) \overline{\tau(\chi_{n_2})} \overline{\tau(\chi_{\delta})}.
 \end{aligned}$$

Using cubic reciprocity and the identity

$$(181) \quad \left(\frac{m}{n}\right)_3 = \overline{\left(\frac{\overline{m}}{\overline{n}}\right)_3}$$

following from the definition of the cubic residue symbol, we get the identities

$$(182) \quad \chi_n(N(m)) \overline{\chi_m}(N(n)) = \left(\frac{N(m)}{n}\right)_3 \overline{\left(\frac{N(n)}{m}\right)_3} = \left(\frac{m}{\overline{n}}\right)_3 = \chi_{\overline{n}}(m)$$

and

$$(183) \quad \overline{\chi_n}(N(m)) \overline{\chi_m}(N(n)) = \overline{\left(\frac{N(m)}{n}\right)_3 \left(\frac{N(n)}{m}\right)_3} = \left(\frac{m}{n}\right)_3 = \chi_n(m),$$

valid for all  $m, n \in \mathbb{Z}[\omega]$  with  $m, n \equiv \pm 1 \pmod{3}$ . We use them to simplify the last line of (180), obtaining

$$(184) \quad \overline{\chi_{n_1}} \chi_{n_2 \delta}(h) \left(\frac{n_2 \delta}{\overline{n_1}}\right)_3 \left(\frac{\delta}{n_2}\right)_3 \tau(\chi_{n_1}) \overline{\tau(\chi_{n_2})} \overline{\tau(\chi_{\delta})}.$$

Now, changing  $n_2 \rightarrow \overline{n_2}$ , the contribution of  $h \neq 0$  to the sum in (176) takes the form

$$\begin{aligned}
 M \sum'_{\substack{N(\Delta) \leq 2Q \\ \Delta \equiv \pm 1 \pmod{3}}} \sum'_{\substack{N(\delta) \leq \frac{2Q}{N(\Delta)} \\ \delta \equiv \pm 1 \pmod{3} \\ (N(\delta), N(\Delta))=1}} \frac{\overline{\tau(\chi_{\delta})}}{N(\delta)} \sum_{l \mid \Delta} \frac{\mu(l)}{l} \overline{\chi_{\delta}}(l) \sum_{h \neq 0} \chi_{\delta}(h) \\
 \times \sum'_{\substack{n_1, n_2 \in \mathbb{Z}[\omega] \\ \frac{Q}{N(\delta \Delta)} < N(n_1), N(n_2) \leq \frac{2Q}{N(\delta \Delta)} \\ n_1, n_2 \equiv \pm 1 \pmod{3} \\ (N(n_1), N(n_2))=1 \\ (n_1 n_2, \delta \overline{\delta} \Delta \overline{\Delta})=1}} \widehat{W}\left(\frac{hM}{lN(n_1 n_2 \delta)}\right) c_{\Delta, \delta, l, h, n_1} c'_{\Delta, \delta, l, h, n_2} \left(\frac{n_1}{n_2}\right)_3,
 \end{aligned}$$

where

$$(185) \quad c_{\Delta, \delta, l, h, n} := \chi_n(l) \overline{\chi_n}(h) \left(\frac{\delta}{\overline{n}}\right)_3 \frac{\tau(\chi_n)}{N(n)} b_{n \Delta \delta}$$

and

$$(186) \quad c'_{\Delta, \delta, l, h, n} := \chi_n(l) \overline{\chi_n}(h) \left(\frac{\delta}{\overline{n}}\right)_3 \frac{\overline{\tau(\chi_n)}}{N(n)} \overline{b_{\overline{n} \Delta \delta}}.$$

Here the presence of  $\left(\frac{n_1}{n_2}\right)_3$  reflects an asymmetry that prevents from using recursion on this problem similar to Heath-Brown's method in [Hea2] and [Hea1]. However, we can estimate the sum over  $n_1$  and  $n_2$  directly using (8) (which itself was proved using recursion). To separate the variables  $n_1, n_2$ , we remove the coprimality condition  $(N(n_1), N(n_2)) = 1$  in the standard way using the Möbius function, and we remove the weight  $\widehat{W}$  using the Mellin transform at essentially no costs (see the treatment of  $\sum_3(M, N)$  in [Hea2]). We further observe that we may freely truncate the sum over  $h$  for

$$(187) \quad |h| \leq \frac{Q^2 l}{N(\delta) N(\Delta)^2 M} (QM)^\varepsilon =: H$$

since  $\widehat{W}$  has rapid decay. Hence, we arrive at sums of the form

$$(188) \quad M \sum_{N(\Delta) \leq 2Q} \sum_{N(\delta) \leq \frac{2Q}{N(\Delta)}} \sum_{l|\Delta} \frac{1}{N(\delta)^{1/2} l} \sum_{|h| \leq H} \left| \sum'_{\substack{N(n_1), N(n_2) \asymp R \\ n_1, n_2 \equiv \pm 1 \pmod{3}}} d_{n_1} d'_{n_2} \left(\frac{n_1}{n_2}\right)_3 \right|,$$

where  $R \ll Q/N(\Delta\delta)$ ,  $d_{n_1}$  and  $d'_{n_2}$  depend on  $\Delta, \delta$  and  $l$ , and

$$(189) \quad |d_n| \ll \left(\frac{N(\delta\Delta)}{Q}\right)^{1/2} |b_{rn}|, \quad |d'_n| \ll \left(\frac{N(\delta\Delta)}{Q}\right)^{1/2} |b_{s\bar{n}}|$$

for some  $r, s \in \mathbb{Z}[\omega]$  with  $N(r), N(s) \asymp Q/R$ . Now, using the Cauchy-Schwarz inequality and the estimate (8) upon noting that this estimate remains valid if the summation conditions  $m, n \equiv 1 \pmod{3}$  herein are replaced by  $m, n \equiv \pm 1 \pmod{3}$ , we bound the inner double sum in (188) by

$$(190) \quad \sum'_{\substack{N(n_1), N(n_2) \asymp R \\ n_1, n_2 \equiv \pm 1 \pmod{3}}} d_{n_1} d'_{n_2} \left(\frac{n_1}{n_2}\right)_3$$

$$(191) \quad \ll \left( \sum'_{\substack{N(n_1) \asymp R \\ n_1 \equiv \pm 1 \pmod{3}}} |d_{n_1}|^2 \right)^{1/2} \left( \sum'_{\substack{N(n_1) \asymp R \\ n_1 \equiv \pm 1 \pmod{3}}} \left| \sum'_{\substack{N(n_2) \asymp R \\ n_2 \equiv \pm 1 \pmod{3}}} d'_{n_2} \left(\frac{n_1}{n_2}\right)_3 \right|^2 \right)^{1/2}$$

$$(192) \quad \ll (QM)^\varepsilon \left(\frac{N(\delta\Delta)}{Q}\right)^{\frac{1}{3}} \sum'_{\substack{N(n) \asymp Q \\ n \equiv \pm 1 \pmod{3}}} |b_n|^2.$$

Inserting this bound into (188) gives a contribution of

$$(193) \quad \ll (QM)^\varepsilon Q^{5/3} \|b_n\|^2.$$

Combining (179) and (193), we deduce that (176) and hence (169) is bounded by

$$(194) \quad \ll (QM)^\varepsilon (M + Q^{5/3}) \|b_n\|^2$$

which implies the desired bound (170).

## 13. COMPLETION OF THE PROOF OF THEOREM 1.4

We obtain the other bounds of Theorem 1.4 by manipulating the bound  $M + Q^{5/3}$  using Hölder's inequality and the relations between norms presented in Sections 9 and 11.

The estimate (172) itself shall turn out to be most useful for small  $Q$  (roughly in the range  $Q \ll N^2$ ). For large  $Q$ , we shall use (172) with  $v = 3$ , giving the bound

$$(195) \quad B_1(Q, M) \ll (QM)^\varepsilon (Q^{11/9} + Q^{2/3}M),$$

as an initial estimate and then derive a new bound improving (172) by employing our results from section 9. Instead of (195), we could use (172) with any  $v \geq 2$  as an initial estimate, but the choice  $v = 3$  shall turn out to be favorable.

From and (142) and (195), it follows that

$$(196) \quad B_3(Q, M) \ll (\log 2Q)^3 Q^{1/2} X^{-1/2} (X^{11/9} + X^{2/3}M)$$

for a suitable  $X$  with  $1 \ll X \ll Q$ . Hence,

$$(197) \quad B_3(Q, M) \ll (\log Q)^3 (Q^{11/9} + Q^{2/3}M).$$

Combining this with (153), we obtain

$$(198) \quad B_4(Q, M) \ll Q + (QM)^{9\varepsilon} QM^{-2} \max \{ K^{11/9} + K^{2/3}M : K \leq M^4 Q^{-1} \} + \\ (QM)^{9\varepsilon} M^6 Q^{-1} \sum_{K \geq M^4/Q} K^{-2-\varepsilon} (K^{11/9} + K^{2/3}M)$$

$$(199) \quad \ll Q + (QM)^{10\varepsilon} (Q^{-2/9} M^{26/9} + Q^{1/3} M^{5/3}),$$

where the sum over  $K$  runs over powers of 2. From this and (144), we deduce that

$$(200) \quad B_3(Q, M) \ll \frac{Q}{\Delta_1} + (QM)^\varepsilon \left( \left( \frac{Q}{\Delta_1} \right)^{-2/9} \cdot \left( \frac{M}{\Delta_2} \right)^{26/9} + \left( \frac{Q}{\Delta_1} \right)^{1/3} \cdot \left( \frac{M}{\Delta_2} \right)^{5/3} \right)$$

for some positive integers  $\Delta_1, \Delta_2$  with  $\Delta_2^2 \geq \Delta_1$ . From this and (143), we deduce that

$$(201) \quad B_1(Q, M) \ll Q + (QM)^\varepsilon (Q^{-2/9} M^{26/9} + Q^{1/3} M^{5/3}).$$

Combining (201) with (126), we deduce that

$$(202) \quad B_1(Q, M) \ll (\tilde{Q}M)^\varepsilon \left( \tilde{Q} + \tilde{Q}^{-2/9} M^{26/9} + \tilde{Q}^{1/3} M^{5/3} \right)$$

if  $\tilde{Q} \geq CQ \log(2QM)$ . We choose  $\tilde{Q} := Q^{1+\varepsilon} + M^{11/5}$ . Then (202) implies that

$$(203) \quad B_1(Q, M) \ll (QM)^\varepsilon (Q + Q^{1/3} M^{5/3} + M^{12/5}).$$

Combining (172) with  $v = 1, 2, 3$  and (203), we obtain our final estimate

$$(204) \quad B_1(Q, M) \ll (QM)^\varepsilon \min \left\{ Q^{5/3} + M, Q^{4/3} + Q^{1/2} M, Q^{11/9} + Q^{2/3} M, Q + Q^{1/3} M^{5/3} + M^{12/5} \right\}.$$

which together with (116) implies Theorem 1.4.  $\square$

Calculating the right-hand side of (4) for various ranges of  $Q$  and  $M$ , we obtain that it is bounded by

$$(205) \quad \ll (QM)^\varepsilon \|a_m\|^2 \cdot \begin{cases} M & \text{if } Q \leq M^{3/5}, \\ Q^{5/3} & \text{if } M^{3/5} < Q \leq M^{6/7}, \\ Q^{1/2} M & \text{if } M^{6/7} < Q \leq M^{6/5}, \\ Q^{4/3} & \text{if } M^{6/5} < Q \leq M^{3/2}, \\ Q^{2/3} M & \text{if } M^{3/2} < Q \leq M^{9/5}, \\ Q^{11/9} & \text{if } M^{9/5} < Q \leq M^{108/55}, \\ M^{12/5} & \text{if } M^{108/55} < Q \leq M^{11/5}, \\ Q^{1/3} M^{5/3} & \text{if } M^{11/5} < Q \leq M^{5/2}, \\ Q & \text{if } M^{5/2} < Q. \end{cases}$$

It can be easily checked that (172) with  $v \geq 4$  does not lead to an improvement of (204).

#### 14. PROOF OF THEOREM 3.1

It remains to prove Theorem 3.1 which gives the basis estimate (31) for  $M_\alpha$  and implies Theorem 1.2. Since all steps are standard, we will be brief.

Using the approximate functional equation and the Cauchy-Schwarz inequality, we estimate the left-hand side of (30) by

$$(206) \quad \sum_{q \leq Q} \sum_{\substack{\chi \pmod{q} \\ \chi^3 = \chi_0}}^{**} |L(1/2 + \alpha, \chi\psi)|^2 \ll S_1 + S_2,$$

where

$$(207) \quad S_1 = \sum_{q \leq Q} \sum_{\substack{\chi \pmod{q} \\ \chi^3 = \chi_0}}^{**} \left| \sum_{m=1}^{\infty} \frac{(\chi\psi)(m)}{m^{\frac{1}{2} + \alpha}} V_\alpha \left( \frac{m}{\sqrt{qk}} \right) \right|^2$$

and

$$(208) \quad S_2 = \sum_{q \leq Q} \sum_{\substack{\chi \pmod{q} \\ \chi^3 = \chi_0}}^{**} \left| \sum_{m=1}^{\infty} \frac{\overline{(\chi\psi)}(m)}{m^{\frac{1}{2} - \alpha}} V_{-\alpha} \left( \frac{m}{\sqrt{qk}} \right) \right|^2.$$

We truncate the sums over  $m$  at  $m = (Qk)^\varepsilon \sqrt{qk}$ , break the summations over  $q$  and  $m$  into dyadic intervals and remove the weights  $V_\alpha$  and  $V_{-\alpha}$  in a standard way using the Mellin



transform. To establish (30), it then suffices to show that

$$(209) \quad \sum_{Q < q \leq 2Q} \sum_{\substack{\chi \pmod{q} \\ \chi^3 = \chi_0}}^* \left| \sum_{M < m \leq 2M} a_m \frac{\chi(m)}{m^{\frac{1}{2} + \alpha}} \right|^2 \\ \ll (QM)^\varepsilon \min \{ Q^{5/3} + M, Q^{4/3} + Q^{1/2}M, Q^{11/9} + Q^{2/3}M, Q + Q^{1/3}M^{5/3} + M^{12/5} \}$$

for any positive  $Q, M$  with  $M \leq (Qk)^{1/2+\varepsilon}$  and any sequence of coefficients  $a_m \ll 1$ . Writing  $m = d^2n$ , where  $n$  is square-free, and using the Cauchy-Schwarz inequality, we find that the sum on the left-hand side of (209) is dominated by

$$(210) \quad (Qk)^\varepsilon \sum_{d \leq \sqrt{2M}} \frac{1}{d} \sum_{Q < q \leq 2Q} \sum_{\substack{\chi \pmod{q} \\ \chi^3 = \chi_0}}^* \left| \sum_{M/d^2 < m \leq 2M/d^2} \mu^2(m) a_{d^2m} \frac{\chi(m)}{m^{\frac{1}{2} + \alpha}} \right|^2.$$

Estimating the inner triple sum in (210) by using Theorem (1.4), we see that (209) holds.  $\square$

## 15. EXTENSION TO SEXTIC CHARACTERS

In this section we prove Theorem 1.5; in addition we also obtain the extension of 3.1 to primitive characters of order dividing 6.

*Proof.* It suffices to prove that Theorem 1.4 holds with  $\chi^6 = \chi_0$  in place of  $\chi^3 = \chi_0$ . Then the correspondingly extended versions of Theorems 1.2 and 3.1 follow in a similar way as before.

We may divide the characters satisfying  $\chi^6 = \chi_0$  into characters of order 2, 3 and 6. The cubic characters were already considered in Theorem 1.4 itself. The contribution of the real characters is bounded using (7), and the term on the right-hand side of (7) is dominated by that on the right-hand side of (4). Thus, for the proof of Theorem 1.5 it suffices to consider the contribution of sextic characters.

We first need to classify these sextic characters. This is very similar to the classification of cubic characters. Again we have precisely two primitive sextic characters of conductor 9 and no primitive cubic character modulo any other power of 3. Furthermore, each primitive sextic character with conductor  $q$  coprime to 3 is of the form  $\psi_n : m \rightarrow \left(\frac{m}{n}\right)_6$  for some  $n \in \mathbb{Z}[\omega]$ ,  $N(n) = q$ ,  $n \equiv 1 \pmod{3}$ ,  $n$  not divisible by any rational primes. We note that  $\psi_n^2 = \chi_n$ .

Now all arguments in Sections 8-11 go through when  $\chi_n$  is replaced by  $\psi_n$ . We note that we need sextic reciprocity only in Section 12 and not before. The sextic reciprocity law can be formulated as follows (see Theorem 7.10 in [Lem]):

$$(211) \quad \left(\frac{m}{n}\right)_6 = (-1)^{\frac{N(m)-1}{2} \frac{N(n)-1}{2}} \left(\frac{n}{m}\right)_6$$

if  $m, n \in \mathbb{Z}[\omega]$  are E-primary. An element  $a + b\omega$  of  $\mathbb{Z}[\omega]$  is E-primary if  $b \equiv 0 \pmod{3}$ ,

$$(212) \quad a + b \equiv 1 \pmod{4} \quad \text{if } 2|b,$$

$$(213) \quad b \equiv 1 \pmod{4} \quad \text{if } 2|a,$$

$$(214) \quad a \equiv 3 \pmod{4} \quad \text{if } 2 \nmid ab.$$

We also have two supplementary laws (see [Lem]). We note that if  $m \equiv 1 \pmod{12}$ , then  $m$  is E-primary. Moreover, if  $m, n \equiv 1 \pmod{12}$ , then sextic reciprocity takes the simple form

$$(215) \quad \left(\frac{m}{n}\right)_6 = \left(\frac{n}{m}\right)_6.$$

Now we explain which modifications of Section 12 are necessary when we work with sextic characters instead of cubic ones. First, the term  $\chi_{n_1} \overline{\chi_{n_2 \delta}}(m)$  occurring in (176) needs to be changed into

$$(216) \quad \psi_{n_1} \overline{\psi_{n_2}}(m) \left(\frac{m}{\delta}\right)_3$$

since

$$(217) \quad \psi_{\delta} \overline{\psi_{\delta}}(m) = \psi_{\delta}^2(m) = \left(\frac{m}{\delta}\right)_6^2 = \left(\frac{m}{\delta}\right)_3.$$

This causes some changes in all what follows in Section 12, but these changes are of minor importance. However, a significant issue is the evaluation of the term  $\psi_{n_1}(N(n_2)) \overline{\psi_{n_2}}(N(n_1))$  which can be extracted by multiplicativity from the last line of the analogue of (180) for sextic characters. In the case when  $n_1, n_2 \equiv 1 \pmod{12}$ , we find that

$$(218) \quad \psi_{n_1}(N(n_2)) \overline{\psi_{n_2}}(N(n_1)) = \left(\frac{N(n_2)}{n_1}\right)_6 \overline{\left(\frac{N(n_1)}{n_2}\right)_6} = \overline{\left(\frac{n_2}{n_1}\right)_3}$$

by using sextic reciprocity and the identity

$$(219) \quad \left(\frac{m}{n}\right)_6 = \overline{\left(\frac{\overline{m}}{\overline{n}}\right)_6}$$

following from the definition of the sextic residue symbol. This is really the only point where we need sextic reciprocity. Everywhere else, it suffices to use the multiplicativity of the residue symbol.

We are now again led to Heath-Brown's result on the large sieve with cubic residue symbols (it is fortuitous that the right hand side of (218) only involves the cubic residue symbol so that we can directly apply (8); in any event, one could probably generalize (8) to sextic characters). Instead of  $\left(\frac{n_2}{n_1}\right)_3$ , we get the conjugate, and under the change  $n_2 \rightarrow \overline{n_2}$ , it turns into  $\left(\frac{n_1}{n_2}\right)_3$ . The estimations in all what follows work similarly as before. The only remaining difficulty is that we need the congruence condition  $n \equiv 1 \pmod{12}$  if we want to apply sextic reciprocity in the form given in (215). We note that in Section 12,  $n$  is assumed to be square-free, coprime to 3, and not divisible by any rational prime and hence not by 2 (which conditions remain unchanged in the case of sextic characters). Therefore,  $n$  is coprime to 12. What we do is to split the sum over  $n$  in our analogue of (169) into sums over the reduced residue classes modulo 12. For each of them, we find  $\rho$  such that  $\rho n \equiv 1 \pmod{12}$ . We now make the change of variables  $n \rightarrow \rho n$ , that is, we write  $\psi_n = \overline{\psi_{\rho}} \psi_{\rho n}$ . This resolves our problem with the additional congruence condition for  $n$ .

The Sections 13 and 14 remain unchanged in the sextic case. We thus obtain an analogue of (4) for sextic characters with the same bound. Combining this with (4) and (7), we arrive at the desired result.  $\square$

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